MATH 7
ASSIGNMENT 11: PASCAL'S TRIANGLE CONTINUED
JAN 16, 2022

Pascal triangle

Recall the Pascal triangle:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

Every entry in this triangle is obtained as the sum of two entries above it. The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by $\binom{n}{k}$. Note that both n and k are counted from 0, not from 1: for example, $\binom{6}{2} = 15$.

These numbers appear in many problems:

$$
\begin{aligned}
\binom{n}{k} &= \text{ The number of paths on the chessboard going } k \text{ units up and } n-k \text{ to the right}\\
&= \text{The number of words that can be written using } k \text{ zeros and } n-k \text{ ones}\\
&= \textbf{The number of ways to choose } k \textbf{ items out of } n \textbf{ (order doesn't matter)}
\end{aligned}
$$

Principle of Counting

In how many ways can you arrange N objects in a sequence (from the 1st to the Nth)?

There are N ways to choose the first one, $N-1$ ways to choose the second object once the first one has been chosen, $N-2$ ways to choose the third object once the first and the second objects have been chosen, an so one. So the Answer is $N!$, which is defined by

$$N! := N(N-1)\dots 1.$$

This is also called P_N, the number of permutations of N elements.

We will see that we can generalize this idea to solve many problems of combinatorics. This idea is sometimes called the *fundamental principle of counting.*

Formula for binomial coefficients

It turns out that there is an explicit formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Compare it with the number of ways of choosing k items out of n when the order matters:

$${}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

For example, there are $5 \cdot 4 = 20$ ways to choose to items out of 5 if the order matters, and $\frac{5\cdot 4}{2} = 10$ if the order doesn't matter.

Homework

1. How many “words” of length 5 one can write using only letters U and R, namely 3 U’s and 2 R’s? What if you have 5 U’s and 3 R’s? [Hint: each such “word” can describe a path on the chessboard, U for up and R for right. . .]

2. How many sequences of 0 and 1 of length 10 are there? sequences of length 10 containing exactly 4 ones? exactly 6 ones?

3. If we toss a coin 10 times, what is the probability that all will will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?

4. In a meeting of 25 people, each must shake hands with each other. How many handshakes are there altogether?

5. (a) An artist has 12 paintings. He needs to choose 4 paintings to include in an art show. How many ways are there of doing this?

(b) The same artist now needs to choose 4 paintings to include in a catalog. How many ways are there to do this? (In the catalog, unlike the show, the order matters).

6. (a) There are 15 students in a soccer club. The coach needs to select 11 of them to form the team for a match against another club. How many possibilities does he have?

(b) There are 15 students in a soccer club. The coach needs to select a goalkeeper and 10 players to form the team for a match against another club. How many possibilities does he have?

(The difference between two parts is that in the first case, the coach needs to select 11 players — no need to specify their positions. In the second part, he needs to select 11 players and specify which of them will be the goalkeeper.)

7. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub — and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 10 steps, he will move

(a) 10 steps north

(b) 10 steps south

(c) 4 steps north

(d) will come back to the starting position

*8 (problems with * are optional) In a party there are 2020 people, which divide into 505 families, each one composed of a father, a mother, and two children. In the party, there are 505 tables, each one with 4 chairs. At a certain point, all the people in the party took a seat randomly, and it was noticed that each table had exactly a father, a mother and two children, though not necessarily from the same family. What is the probability that each table has the mambers of the same family?