MATH 7
ASSIGNMENT 12: BINOMIAL PROBABILITIES
JAN 23, 2022

Today we talk about two very useful applications of binomial numbers, the coefficients appearing in Pascal's triangle.

Binomial Theorem

Binomial coefficients that appear in Pascal Triangle have an important application in algebra. They allow us to find an expansion of expressions such that $(a+b)^2, (a+b)^3, (a+b)^4, (a+b)^5, (a+b)^6$?

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$$

Notice that the coefficients correspond to a row in Pascal's Triangle. The coefficient of the term $a^{n-k}b^k$ is $\binom{n}{k}$.

Square of a Sum, Difference of Squares

Recall the following formulas. They are all special cases of Binomial Theorem!

- $(a+b)^2 = a^2 + 2ab + b^2$ (square of a sum)
- $(a-b)^2 = a^2 - 2ab + b^2$ (square of a difference)
- $a^2 - b^2 = (a+b)(a-b)$ (difference of squares)
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (cube of a sum)
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ (cube of a difference)

Binomial Probabilities

These numbers are also useful in calculating probabilities. Imagine that we have some event that happens with probability p ("success") and does not happen with probability $q = 1-p$ ("failure"). Then the probability of getting k successes in n trials is

$$P(k \text{ successes in } n \text{ trials }) = \binom{n}{k}p^k q^{n-k}, \text{ where}$$

- p — probability of success in one try;
- $q = 1-p$ — probability of failure in one try;
- n — number of trials;
- k — number of successes;
- $n-k$ — number of failures.

Example: You roll a dice 100 times. What is the probability of getting a 6 exactly 20 times?
Solution: Here we have: $n = 100$, $k = 20$, $p = 1/6$, $q = 5/6$. Then

$$P = \binom{100}{20} \cdot \left(\frac{1}{6}\right)^{20} \left(\frac{5}{6}\right)^{80}$$

Homework

1. Expand $(a+1)^7$
2. Expand $(x+2y)^4$
3. A test consists of 10 multiple choice questions with five choices for each question. As an experiment, you GUESS on each and every answer without even reading the questions. What is the probability of getting exactly 6 questions correct on this test?
4. In a certain city, 30% of the voters prefer candidate A over the others. If 10 voters are chosen randomly, what is the probability that 30% of them prefer candidate A?
5. In a heads or tails game, a try consists of tossing a coin three consecutive times. The "try" is considered a success if one gets strictly more heads than tails. What is the probability that one will succeed in the first two tries?
6. How many words one can get by permuting letters of the word "tiger"? of the word "rabbit"? of the word "mammoth"?
7. 5 friends go to a ramen restaurant for dinner. In this restaurant there are 17 different types of ramen.

(a) If each one orders one ramen, how many possible combinations are there?
(b) If each one orders one ramen so that they all choose differently, how many possible combinations are there?
(c) In how many ways can one choose 5 types of ramen (for oneself) from the menu?

Extra Problems (Optional)

1. What is the coefficient of the term x^9 in $(x^2 - 2x)^6$?

2. Expand $\left(x + \dfrac{1}{x}\right)^3$

3. Use the Binomial Theorem to prove the identity:
$\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n-1} + \binom{n}{n} = 2^n$

4. A box contains 1 black ball and 9 white balls. A second box also contains 10 balls, x of which are black and the others are white. Superman takes one ball of each box randomly. Batman puts the balls of both boxes together in a large box and then takes two balls randomly. What is the minimum value of x such that Batman is more likely to get two black balls than Superman?