MATH 10
ASSIGNMENT 13: ARITHMETIC AND GEOMETRIC SEQUENCES
JANUARY 30, 2022

Arithmetic Sequence

Definition. A sequence of numbers is an *arithmetic sequence* or *arithmetic progression* if the difference between consecutive terms is the same number, the *common difference* or d.

For example, the sequence $1, 5, 9, 13, 17, \dots$ is an arithmetic sequence because the difference between consecutive terms is $d = 4$. Note that one can always find the n-th term if the 1st term and d are given. For example, what is the 100-th term of this sequence?

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + d = 1 + 4 = 5 \\ &\dots \\ a_{100} &= a_1 + 99d = 1 + 99 \times 4 = 397. \end{aligned}$$

Thus the general form of the n-th term,

$$a_n = a_1 + (n-1)d.$$

Another useful formula is the sum of the first n terms,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = n \times \frac{a_1 + a_n}{2}.$$

The best is to remember the proof, because this trick comes up every now and then: first write the sum in 2 ways, in increasing and decreasing order:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ S_n &= a_n + a_{n-1} + \cdots + a_1, \end{aligned}$$

adding these two expressions up and noticing that $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$ we get

$$\begin{aligned} 2S_n &= (a_1 + a_n) \times n \\ S_n &= n \times \frac{a_1 + a_n}{2}. \end{aligned}$$

Geometric Sequence

Definition. A sequence of numbers is a *geometric sequence* or *geometric progression* if if the next number in the sequence is the current number times a fixed constant called the *common ratio* or q.

The sequence $6, 12, 24, 48, \dots$ is a geometric sequence because the next number is obtained from the previous by multiplication by $q = 2$. For example, let us find a_{10} for the sequence above.

$$\begin{aligned} a_1 &= 6 \\ a_2 &= a_1 q = 6 \cdot 2 = 12 \\ &\dots \\ a_{10} &= a_1 q^9 = 6 \cdot 2^9 = 6 \cdot 512 = 3072, \end{aligned}$$

so that the general form is

$$a_n = a_1 q^{n-1}$$

Another useful formula is the sum of the first n terms,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{a_1(1 - q^n)}{1 - q}$$

Again the proof is based on a *very useful* trick: write the sum and multiply it by q:

$$S_n = a_1 + a_2 + \cdots + a_n,$$
$$qS_n = qa_1 + qa_2 + \cdots + qa_n.$$

Now notice that $qa_1 = a_2, \ldots qa_2 = a_3, \ldots, qa_n = a_{n+1}$, etc, so we have

$$S_n = a_1 + a_2 + \cdots + a_n,$$
$$qS_n = a_2 + a_3 + \cdots + a_{n+1}.$$

Subtracting the second equality from the first and canceling out the terms we get

$$\begin{aligned} S_n - qS_n &= (a_1 - a_{n+1}) \\ \Rightarrow S_n(1-q) &= (a_1 - a_1 q^n) \\ \Rightarrow S_n(1-q) &= a_1(1-q^n), \end{aligned}$$

from which we get the formula above.

Infinite Sum

If $0 < q < 1$, then the sum of the geometric progression approaches a fixed number, which we can call a *sum of an infinite geometric progression*, or just an *infinite sum*. For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

The formula for the infinite sum is

$$S = \frac{a_1}{1-q},$$

which follows from the formula for S_n since, as n increases, $q^n \to 0$ if $0 < q < 1$.

Homework

1. Write the first 5 terms of an arithmetic sequence if $a_1 = 7$ and $d = 2$.

2. In the arithmetic progression $5, 17, 29, 41, \ldots$ what term has a value of 497?

3. Find the sum of the first 10 terms for the series: $4, 7, 10, 13, \ldots$

4. In a given arithmetic progression, the first term is 6, and the 87-th term is 178. Find the common difference of this arithmetic progression, and give the value of the first five terms.

5. There are 25 trees at equal distances of 5 meters in a line with a well, the distance of the well from the nearest tree being 10 meters. A gardener waters all trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.

6. Prove that, in an arithmetic sequence, any term is the arithmetic mean of its neighbors:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

7. Write out the first four terms of each of the following geometric sequence, given the first term $b_1 = 27$ and the common ratio $q = -\frac{1}{3}$.

8. What are the first two terms of the geometric progression $a_1, a_2, 24, 36, 54, \ldots$?

9. Prove that, in a geometric progression, any term is the *geometric mean* of its neighbors,

$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}.$$

10. Calculate the following sums:

(a) $1 + 3 + 9 + 27 + 81 + 243.$

(b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}.$

(c) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots$

11. Musicians use special notations for notes, i.e. sound frequencies. Namely, they go as follows:

$$\ldots, \text{A}, \text{A}\sharp, \text{B}, \text{C}, \text{C}\sharp, \text{D}, \text{D}\sharp, \text{E}, \text{F}, \text{F}\sharp, \text{G}, \text{G}\sharp, \text{A}, \text{A}\sharp, \ldots$$

The interval between two notes in this list is called a *halftone*; the interval between A and the next A (or B and next B, etc.) is called an *octave*. Thus, one octave is 12 halftones. (If you have never seen it, read the description of how it works in Wikipedia.) It turns out that the frequencies of the notes above form a geometric (not an arithmetic!!) sequence: if the frequency, say, of A in one octave is 440 hz, then the frequency of A$\sharp$ is $440r$, frequency of B is $440r^2$, and so on.

(a) It is known that moving by one octave doubles the frequency: if the frequency of A in one octave is 440 hz, then the frequency of A in the next octave is $2 \times 440 = 880$ hz. Based on that, can you find the common ratio r of this geometric sequence?

(b) Using the calculator, find the ratio of frequencies of A and E (such an interval is called a *fifth*). How close is it to $3 : 2$?

(Historic reference: the above convention for note frequencies is known as *equal temperament* and was first invented around 1585. However, it was not universally adopted until the beginning of 19th century. One of the early adopters of this tuning method was J.-S. Bach, who composed in 1722-1742 a collection of 48 piano pieces for so tuned instruments, called Well-Tempered Clavier. Find them and enjoy!)

12. Try to use a similar logic as the one used to prove the formula for the sum of the geometric progression to calculate

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots.}}}$$

Extra Problems (Optional)

1. Find the sum of the first 1000 odd numbers.

2. The 3-rd term of the arithmetic progression is equal to 1. The 10-th term of it is three times as much as the 6-th term. Find the first term and the common difference.

3. The sum of the first 20 terms of an arithmetic progression is 200, and the sum of the next 20 terms is -200. Find the sum of the first hundred terms of the progression.

4. Prove that one can find the common difference d using any two given terms a_m and a_n by

$$d = \frac{a_m - a_n}{m - n}.$$