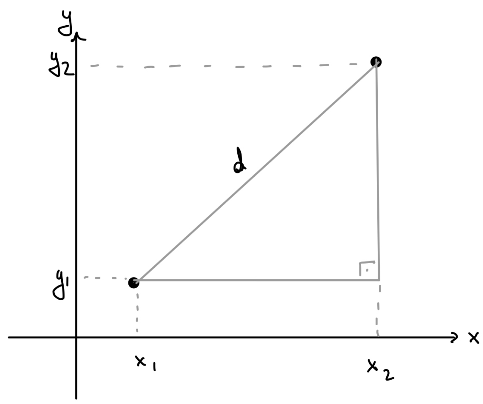
MATH 7
ASSIGNMENT 17: ISOMETRIES OF THE PLANE
MAR 20, 2022

Today we will work on a few problems with vectors from the previous assignment and learn about isometries.

Isometries

Let us use a fixed system of coordinates, so that each point corresponds to a pair of real numbers $(x, y) \in \mathbb{R}^2$. Then, from Pythagoras's theorem, we have the distance between two points:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



We call an *isometry of the plane* a function $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto f(x, y)$ such that $d(f(x_1, y_1), f(x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$ for any pair of points (x_1, y_1) and (x_2, y_2). We say that the transformation f *preserves distances*. There are three types of such transformations: rotations, reflections and translations.

Rotations

Here we will always express rotations around the origin, by an angle ϕ which is measured from the x axis in the counterclockwise direction. As we show in the class, a point of coordinates (x, y) is rotated into the point

$$R_\phi(x, y) = (\cos\phi \cdot x - \sin\phi \cdot y, \sin\phi \cdot x + \cos\phi \cdot y).$$

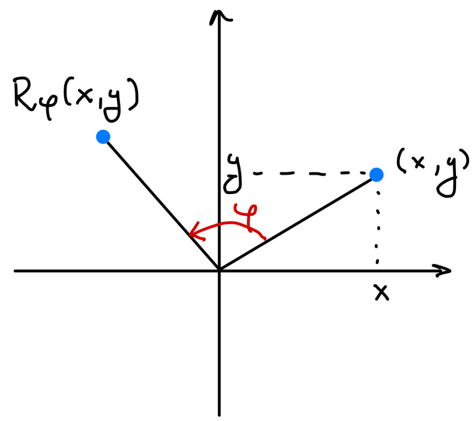
Reflections

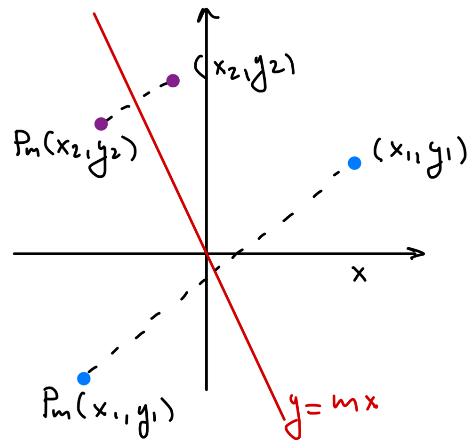
Another set of isometries of the plane is given by reflections with respect to a line that passes through the origin. We will denote the reflection with respect to the line $y = mx$ by P_m. In particular, the reflections with respect to the x and y axes are

$$P_0(x, y) = (x, -y)$$

and

$$P_\infty(x, y) = (-x, y)$$

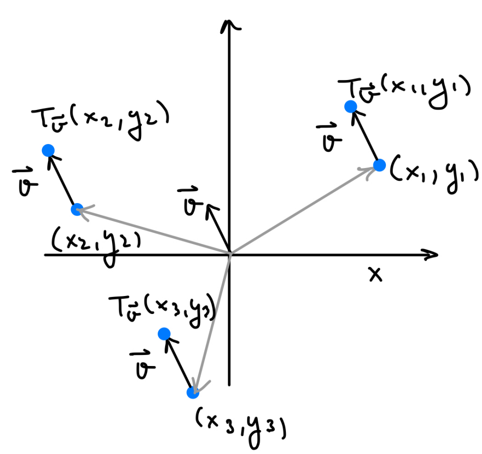




Translations

The third type of isometries are translations. A good way of thinking of them is to consider each point of the plane (x, y) and a vector. Then a $T_{\vec{v}}$ by vector $\vec{v}$ is given by adding $\vec{v}$ to all point vectors (x, y):

$$T_{\vec{v}}(x, y) = (x + v_x, y + v_y)$$



Composition of Isometries

The composition $f_1 \circ f_2$ of two isometries f_1 and f_2 is an isometry:

$$d(f_1 \circ f_2(x,y)) = d(f_1(f_2(x,y))) = d(f_2(x,y)) = d(x,y)$$

Inded, the transformations defined above can be composed in interesting ways, as you will see in the homework.

Homework

1. Let $A = (1,0)$, $B = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $C = (-\frac{1}{2}, \frac{-\sqrt{3}}{2})$
   (a) Sketch these three points in the coordinate plane
   (b) Calculate $d(A,B)$, $d(B,C)$ and $d(C,A)$. Use this to argue that ΔABC is an equilateral triangle.
   (c) What is the smallest angle ϕ such that the rotation R_ϕ preserves the triangle ΔABC. [Hint: consider the rotation which permutes the vertices by: $A \mapsto B$, $B \mapsto C$ and $C \mapsto A$.]
   (d) Use the angle ϕ from the previous item in the formula $R_\phi(x,y) = (\cos\phi \cdot x - \sin\phi \cdot y, \sin\phi \cdot x + \cos\phi \cdot y)$ to show that $R_\phi(A) = B$, $R_\phi(B) = C$ and $R_\phi(C) = A$.
   (e) Show that the reflection $P_0(x,y) = (x,-y)$ also preserves the triangle.
   (f) Can you think of other isometries which preserve the triangle (simply permute the vertices)? The set of all of these form the *Dihedral group* of the triangle.

2. Repeat all of problem 1 for the square formed by the vertices $A = (1,0)$, $B = (0,1)$, $C = (-1,0)$ and $D = (0,-1)$.

3. Prove by explicit calculation that $d(P_0(x_1,y_1), P_0(x_2,y_2)) = d((x_1,y_1),(x_2,y_2))$ for any pair of points (x_1,y_1) and (x_2,y_2)

4. Prove by explicit calculation that $d(T_{\vec{v}}(x_1,y_1), T_{\vec{v}}(x_2,y_2)) = d((x_1,y_1),(x_2,y_2))$ for any pair of points (x_1,y_1) and (x_2,y_2) and for any vector $\vec{v}$.

Optional

1. Prove by explicit calculation that $d(R_\phi(x_1,y_1), R_\phi(x_2,y_2)) = d((x_1,y_1),(x_2,y_2))$ for any pair of points (x_1,y_1) and (x_2,y_2) and for any angle ϕ.

***2.** Can you think of a way to express a reflection P_m with respect to a line of arbitrary slope m by composing P_0 with rotations?