MATH 7
ASSIGNMENT 19: MATHEMATICAL INDUCTION

APR 3, 2022

Mathematical Induction

Consider the following statement: the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. This is true for all n. How do we prove it?

One important method to prove these statements is that of *mathematical induction.*First we show that it is true for the smallest possible value of n. Indeed, for $n = 1$, we can check directly:

$$1 = \frac{1(1+1)}{2}.$$

This is called the *initial step*. Then we assume that it is true for a certain value $n = k$: the *inductive assumption*. In this case, it means assuming that

$$1 + 2 + ... + k = \frac{k(k+1)}{2}.$$

Finally, we show that because it is true for $n = k$ then it is true for $n = k+1$. This final step is called the *inductive step*. This is the hardest step of the proof, and it makes use of the inductive assumption. Here,

$$1 + 2 + ... + k + k + 1 = \frac{k(k+1)}{2} + k + 1 = \frac{(k+2)(k+1)}{2} = \frac{(k+1)((k+1)+1)}{2},$$

and this concludes the proof.

Let us think why we just proved the equality for all n. We proved directly that it is true for $n = 1$. And we showed that if it is true for $n = k$ then it is also true for $n = k+1$. Therefore it is also true for $n = 2$. Similarly, it follows that the statement is true for $n = 3$ and so on! To summarize, the three steps in a proof by induction are:

1. Prove the initial case (like $n = 1$)
2. Write down the statement for $n = k$ and assume it is true
3. Show that it follows from the previous step that the statement is true for $n = k+1$

Homework

In all problems, use induction to prove the statements

1. Show that the sum of the first n odd positive integers is n^2: $1 + 3 + 5 + ... + (2n-1) = n^2$.

2. Prove the formula for the sum of terms in a geometric sequence:

$$1 + r + r^2 + r^3 + ... + r^n = \frac{1 - r^{n+1}}{1 - r}$$

3. Prove the formula for the sum of squares:

$$1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

4. Prove that $n^3 + 2n$ is divisible by 3 for any integer n

5. Prove that $2^n + 1$ is divisible by 3 for all odd integers n

Optional

1. Prove that $n^2 - 1$ is divisible by 8 for any odd integer n
2. Prove that a convex n-gon (a polygon with n sides) has $\frac{n(n-3)}{2}$ diagonals.