MATH 7
ASSIGNMENT 20: FIBONACCI NUMBERS
APR 10, 2022

Fibonacci Numbers

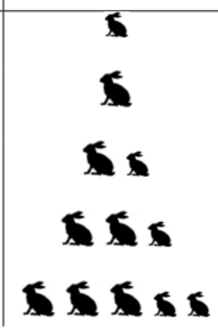
The Fibonacci numbers are a sequence defined by $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, ...$$

This simple arithmetic sequence has sparked the interest of mathematicians throughout history and across the world since ancient times! We will see today a few properties of these numbers.

Homework

1. Somebody buys a pair of rabbits and places them in a pen. The nature of rabbits is such that each month pair of rabbits gives birth to another pair, and they start reproducing once they are 2 months old. How many pairs of rabbits will this person have after one year (assuming that no rabbits die)? [This story is attributed to Leonardo of Pisa, also called Fibonacci, 1202]



month		number of pairs
1		1
2		1
3		2
4		3
5		5

2. Use mathematical induction to prove that $F_1 + F_2 + ... + F_n = F_{n+2} - 1$ for all $n \geq 1$.

3. Use mathematical induction to prove that $F_2 + F_4 + ... + F_{2n} = F_{2n+1} - 1$ for all $n \geq 1$.

4. Use mathematical induction to prove that $F_1 - F_2 + F_3 - F_4 + ... + (-1)^n F_{n+1} = (-1)^n F_n + 1$ for all $n \geq 1$.

5. Here we derive a general formula for the terms of the Fibonacci sequence F_n

(a) Suppose that the terms are some type of geometric sequence, $F_n = aq^n$. Then substituting this guess in the recursion relation $F_{n+1} = F_n + F_{n-1}$ find the two possible values q_1 and q_2 for the common ratio

(b) Now use these two values and suppose that $F_n = aq_1^n + bq_2^n$. Use the first few terms of the Fibonacci sequence to find a and b.

(c) Use mathematical induction to prove that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

for all n. The number $\Phi = \frac{1+\sqrt{5}}{2}$ is called the *golden ratio* and has a long of history too!

(d) If n is really large, can you guess the approximate value of the ratio F_{n+1}/F_n?

Optional

1. Use mathematical induction to prove that $F_1^2 + F_2^2 + F_3^2 + F_4^2 + ... + F_n^2 = F_n F_{n+1}$ for all $n \geq 1$.

2. (a) Which Fibonacci numbers are even? Can you find a pattern?

(b) Prove your claim about which Fibonacci numbers are even.

3. Consider the rectangle with sides 1 and Φ. Show that if we cut from it a 1×1 square, then the remaining rectangle will again have proportions $1 : \Phi$.