MATH 7
ASSIGNMENT 21: REVIEW TOURNAMENT - QUADRATIC POLYNOMIALS
APR 24, 2022

Today's battle will review all the main points we covered on dealing with quadratic polynomials.

Homework

1. Factor (i.e., write as a product) the following expressions. For example, $x^2 - 1 = (x-1)(x+1)$.
 (a) $9x^2 - 25$
 (b) $(x-2)^2 - 10(x-2) + 25$
 (c) $x^3 - x^2 + \frac{1}{3}x - \frac{1}{27}$
 (d) $x^4 + 4$ [Hint: add and then subtract $4x^2$.]
 (e) $x^3 - 27$
 Then use the factorized forms to find all real roots of each polynomial in (a) to (e).

2. Let x_1, x_2 be roots of the equation $ax^2 + bx + c = 0$. Using Vieta's formulas, express in terms of a, b and c:
 (a) $x_1 + x_2$
 (b) $x_1 x_2$
 (c) $x_1^2 + x_2^2$
 (d) $(x_1 - x_2)^2$
 (e) $\frac{1}{x_1} + \frac{1}{x_2}$
 (f) $x_1^3 + x_2^3$
 Then use these formulas to numerically evaluate (a) through (f) for the case $x^2 + 5x - 7 = 0$.

3. Derive the general formulas to solve the quadratic equation $ax^2 + bx + c = 0$ by following these steps:
 (a) Suppose it is possible to factor the polynomial $ax^2 + bx + c$ into the form $a(x - x_1)(x - x_2)$ for some numbers x_1 and x_2. Then prove that x_1 and x_2 are the roots of this polynomial.
 (b) Using the factorization in the previous part, prove Vieta's formulas

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}.$$

 (c) Define the average $x_M = \frac{x_1 + x_2}{2}$ and the distance $d = \frac{|x_1 - x_2|}{2}$. Use Vieta's formulas to calculate x_M and d in terms of the coefficients a, b and c. [Hint: use your techniques from problem 2]
 (d) Now express the roots x_1 and x_2 in terms of x_M and d. Finally, use this to derive that

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

4. For each polynomial $ax^2 + bx + c$ bellow, determine how many roots it has. Find the roots. Then study its sign (by factoring the polynomial). And finally use all that knowledge to sketch the graph $y = ax^2 + bx + c$.
 (a) $x^2 - 5x + 5$
 (b) $x^2 - 5x - 14$
 (c) $-x^2 + 11x - 28$
 (d) $x^2 - x - 1$
 (e) $x^2 + 2x - 3$
 (f) $-x^2 + 6x - 9$

5. (a) Show that, if $y = ax^2 + bx + c$, then

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right], \text{ where } D = b^2 - 4ac$$

 (b) Now show that, if $a > 0$, the polynomial $y = ax^2 + bx + c$ has a minimum at $x = -\frac{b}{2a}$, and the value there is $-\frac{D}{4a}$. What about when $a < 0$?
 (c) Of all the rectangles with perimeter 4, which one has the largest area?

6. A few university students devised a rocket prototype, but it failed. When launched, the rocket followed a trajectory $h = -d^2 + 101x - 100$, where h and d denote the height and d denotes the horizontal displacement on the ground from the launching point at each instant (both are measured in meters).

(a) What is the shape of the trajectory?

(b) Where did the rocket fall?

(c) For which values of d was the rocket at a height bigger than 2000?