MATH 7
ASSIGNMENT 23: REVIEW TOURNAMENT - INDUCTION: PROGRESSIONS, PASCAL AND FIBONACCI

MAY 8, 2022

Today's battle will review sequences and counting problems which can be proven using induction.

Homework

1. (a) Give an explanation for the formula

$$1 + 2 + 3 + ... + 98 + 99 + 100 = 50 \times 101,$$

then generalize this reasoning to show that, if $a_n = a_1 + (n-1)d$ is an arithmetic progression with initial term a_1 and common difference d, then

$$a_1 + a_2 + ... + a_{n-1} + a_n = \frac{n(a_1 + a_n)}{2}.$$

(b) Now prove the previous equation (part (b)) by induction in n.

(c) Now consider a geometric sequence $a_n = a_1 q^{n-1}$ with initial term a_1 and common ratio q. Prove by induction in n that

$$a_1 + a_2 + ... + a_{n-1} + a_n = \frac{a_1(1-q^n)}{1-q}.$$

(d) Now prove by induction in n the formula for the sum of squares: if $a_n = n^2$, then

$$a_1 + a_2 + ... + a_{n-1} + a_n = \frac{n(n+1)(2n+1)}{6}.$$

2. Recall the Pascal triangle:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

It is defined by the following rule: on each row, the first entry is 1, the last entry is 1, and the intermediate ones are the sum of the two entries above it. For example, in the third row from the top, $2 = 1 + 1$ and in the fourth row $3 = 1 + 2$. The k-th entry in n-th row is denoted by $\binom{n}{k}$. Note that both n and k are counted from 0, not from 1: for example, $\binom{6}{2} = 15$.

(a) There is a general formula for the entries of Pascal's triangle:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n! = n \times (n-1) \times ... \times 1$ and $0! = 1$. Prove this formula by induction in the row position n. [Hint: the main step of your proof should be the identity $\binom{n-1}{k} + \binom{n-1}{k+1} = \binom{n}{k+1}$.]

(b) Now prove the following formula for the sum of all entries in a given row,

$$\binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n,$$

by induction in n. [Hint: use the identity $\binom{n-1}{k} + \binom{n-1}{k+1} = \binom{n}{k+1}$.]

3. The Fibonacci numbers are a sequence defined by $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, ...$$

(a) Use mathematical induction to prove that $F_1 + F_2 + ... + F_n = F_{n+2} - 1$ for all $n \geq 1$.
(b) Which Fibonacci numbers are even? Can you find a pattern?
(c) Prove your claim about which Fibonacci numbers are even.