MATH 7

ASSIGNMENT 24: INTRODUCTION TO CALCULUS - LINEAR APPROXIMATIONS

MAY 15, 2022

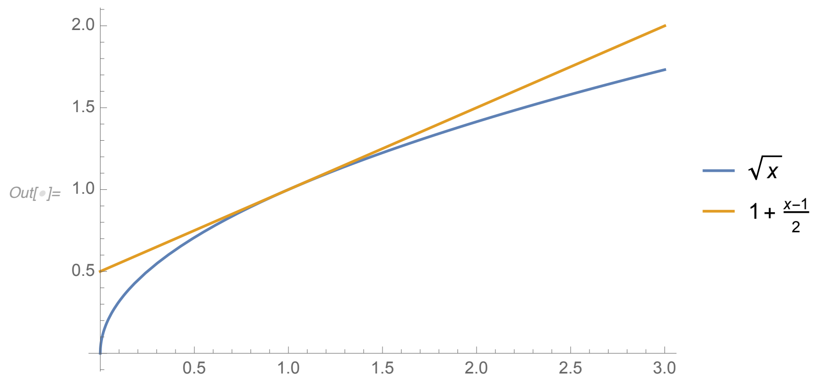
Today we take a break from the review to learn something different. Consider a general function $f(x)$. Sometimes it is difficult to calculate its value in an arbitrary point x, but approximations might be useful. For example, what is the approximate value of $\sqrt{10}$? Or of $17^{\frac{3}{2}}$? or of $\sin 10^{\circ}$?

Linear Approximation

The idea of the linear approximation is to estimate the value of $f(x)$ around point $x = a$ by using its value at that point $f(a)$ and the slope of the tangent at point a, $f'(a)$:

$$f(x) \approx f(a) + f'(a)(x - a). \tag{1}$$

These pictures show the linear approximation of the square root around $a = 1$:



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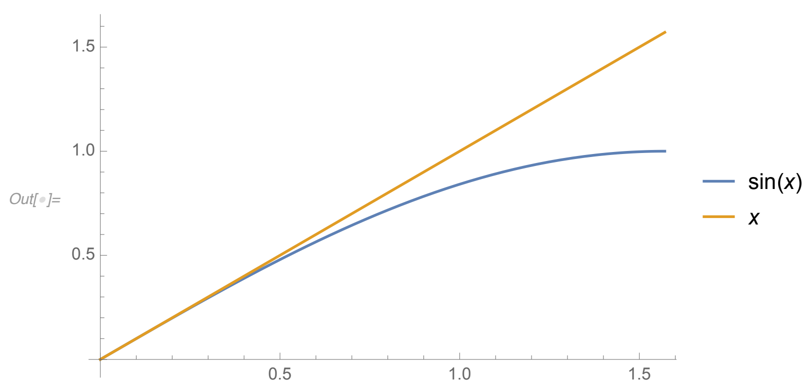
and of the sine around $a = 0$:



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Derivative

The hard part of the linear approximation is knowing the value of the slope of the tangent line to the graph of the function $f(x)$. This is given by the derivative and is denoted by $f(x)'$ or $f'(x)$, and you will learn how to compute it when you take calculus. Today we will simply use a few given results for the derivative:

$$(x^n)' = nx^{n-1},$$
$$(\sin(x))' = \cos(x),$$
$$(\cos(x))' = -\sin(x).$$

Applications

Here are a few examples of how to compute approximate values using the above prescription. First of all, we have the useful formula

$$(1+x)^n \approx 1 + nx,$$

if x is small. This is a consequence of applying linear approximation (1) to the function $f(x) = x^n$ close to 1:

$$x^n = f(x) \approx f(1) + f'(1)(x-1) = 1^n + n1^{n-1}(x-1) = 1 + n(x-1).$$

And it can be used to approximate roots, for example:

$$\sqrt{10} = \sqrt{9+1} = \sqrt{9}\sqrt{1+\frac{1}{9}} = 3\left(1+\frac{1}{9}\right)^{\frac{1}{2}} \approx 3\left(1+\frac{1}{2}\frac{1}{9}\right) = 3 + \frac{1}{6} \approx 3.16,$$

which is quite close: $\sqrt{10} \approx 3.16228$. Note that the approximation is especially simple close to $x = 1$ and that is why we factored out the 9. Another useful approximation is that

$$\sin(x) \approx x,$$

for x small and *in radians*. This comes from, taking $f(x) = \sin(x)$ and expanding close to $a = 0$,

$$\sin(x) = f(x) \approx f(0) + f'(0)(x-0) = \sin(0) + \cos(0)(x-0) = x,$$

and it can be applied in the following way:

$$\sin(10^\circ) = \sin\left(10\frac{2\pi}{360}\right) = \sin\left(\frac{\pi}{36}\right) \approx \frac{\pi}{36} \approx \frac{1}{12} \approx 0.08,$$

which is not bad: $\sin(10^\circ) \approx 0.0872$.

Homework

1. In each case, use the linear approximation to estimate the expression:
 (a) $\sqrt{26}$
 (b) $\sqrt[3]{30}$
 (c) $\sqrt{7}$
 (d) $\sqrt[5]{37}$
 (e) $200^{\frac{3}{2}}$
 (f) $150^{\frac{2}{3}}$
 Then verify the values in a calculator and compare. Which approximations worked best? Why?

2. In each case, use the linear approximation to estimate the expression. Note that you should approximate from an angle as close as possible to the one you wish you evaluate!
 (a) $\sin\frac{\pi}{8}$
 (b) $\cos 50^\circ$
 (c) $\sin 32^\circ$
 (d) $\sin -5^\circ$
 (e) $\sin\frac{9\pi}{8}$
 Then verify the values in a calculator and compare. Which approximations worked best? Why?

3. Without using a calculator, estimate:
 (a) The size of the diagonal of a ruler. The ruler is 30cm long and 2cm wide.
 (b) At a certain moment, an astronomer observed Venus in the sky making a 40° angle with the Sun, and from the observational data estimated that Venus was a distance of about 90 million km away. Since the Sun is about 150 million km away from us, can you estimate the distance Venus is from the Sun? Then look up the actual value and discuss how you could improve your estimate.

***4.** (Optional). This exercise will justify the formula $(x^n)' = nx^{n-1}$ for the slope of the line tangent to the curve $y = x^n$.

(a) Consider the graph $y = x^n$. Write an expression for the slope m of the line which crosses this graph at points $(a, y(a))$ and $(b, y(b))$.
(b) Now adapt this formula for the case when $b = a + \delta$. δ is the Greek letter delta and is used to denote small quantities. For δ small, it means we take two nearby points in the curve $y = x^n$.
(c) Remember the binomial theorem: $(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + ... + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$ and apply it to the expression for $y(a + \delta)$.
(d) Finally, use the above expressions to show that $m = na^{n-1}$ plus terms which involve at least one factor of δ, so that they all vanish as δ becomes smaller and smaller. Since δ small means that the points are close together this means that we found the slope of the tangent at point $(a, y(a))$!