MATH 8
HANDOUT 10: LOGIC PROOFS

Basic logic operations

$\neg$: NOT (for example, NOT A): true if A is false, and false if A is true.
$\wedge$: AND (for example A AND B): true if both A, B are true, and false otherwise.
$\vee$: OR (for example A OR B): true if at least one of A, B is true, and false otherwise.
$\implies$: IF... THEN (for example $A \implies B$: "if A, then B): if A is false, automatically true; if A is true, it is true only when B is true

Proofs

A proof is a sequence of statements, starting with given ones and ending in a statement which we want to prove, and such that each statement in the sequence logically follows from the previous. In the simple case when all our statements can be written as combinations of the same elementary statements (which we can denote by letters $A, B, \dots$), using logical operations, it means

> For any combinations of values of letters $A, B, \dots$ which makes the previous statements true, the next statement is also true.

Here are some commonly used logic laws (all of them can be proved using truth tables):

- Given $A \implies B$ and A, we can conclude B (Modus Ponens)
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! it only means that **if** A is true, then so is C.]
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ (De Morgan Law)
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ (law of contrapositive)

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

Common methods of proof

Conditional proof: To prove $A \implies B$, **assume** that A is true; derive B using this assumption.

Proof by contradiction: To prove A, assume that A is **false** and derive a contradiction (i.e., something which is always false – e.g. $B \wedge \neg B$).

Combination of the above: To prove $A \implies B$, assume that A is true and that B is false and then derive a contradiction.

Homework

1. The following statement is sometimes written on highway trucks:

> If you can't see my windows, I can't see you.

Can you write an equivalent statement without using word "not" (or its variations such as "can't").

2. Which of the statements below is equivalent to the statement "If you do not work hard, you fail the course":

(a) If you work hard, you do not fail the course.
(b) If you failed the course, you did not work hard.
(c) If you did not fail the course, you worked hard.

3. Consider the following statement:

You can't be happy unless you have a clear conscience.

Can you rewrite it using the usual logic operations such as $\wedge$, $\vee$, $\implies$? Use letter H for "you are happy" and C for "you have a clear conscience".

Note: proving this statement is not part of the assignment :).

4. Use proof by contradiction to prove the following statement:

If the square of an integer number n is even, then n itself is even.

You can use without proof that every integer number is either even (i.e., can be written in the form $n = 2k$, with integre k) or odd (i.e., can be written as $n = 2k + 1$, with integer k).

5. Prove by contradiction that there does not exist a smallest positive real number.

6. Prove that there do not exist integers m, n such that 4m+2n=5

7. $a < b$ is a sufficient condition for $4ab < (a + b)^2$. Is this condition also necessary?

8. Prove the following logical equivalence:

$$\neg (p \implies q) \iff (p \wedge \neg q)$$

9. (a) Prove the following logical equivalence:

$$p \vee (q \vee r) \iff (p \vee q) \vee r$$

Thus the logical or $\vee$ is said to be *associative*.

(b) Prove that the logical and $\wedge$ is also associative.

(c) **Disprove** the following logical equivalence:

$$(p \implies (q \implies r)) \iff ((p \implies q) \implies r)$$

[Hint: which of the statements is true if p, q, and r are all false?] Conclude that the logical implication $\implies$ is not associative.

10. Given logical statements m, p, q, let a denote the combined statement $(m \wedge p) \vee (\neg m \wedge q)$. In other words, $a \iff ((m \wedge p) \vee (\neg m \wedge q))$. Prove the following:

(a) If m is true, then $a \iff p$

(b) If m is false, then $a \iff q$