MATH 8: HANDOUT 14

1. Triangles

Theorem 8. *Given any three points A, B, C, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a* triangle*, written $\triangle ABC$. The three respective angles are called the triangle's* interior angles.*)*

2. Congruence

It will be helpful, in general, to have a way of comparing geometric objects to tell whether they are the same. We will build up such a notion and call it congruence of objects. To begin, we define congruence of angles and congruence of line segments (note that an angle cannot be congruent to a line segment; the objects have to be the same type).

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong CD$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \implies \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

3. Congruence of Triangles

Triangles consist of six pieces (three line segments and three angles), but some notion of constancy of shape in triangles is important in our geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we can take them as axioms and continue.

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

4. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem 9 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

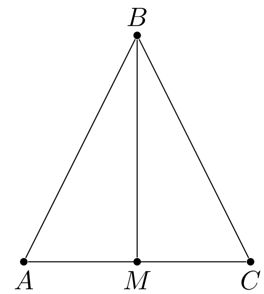
Proof. Assume that $\triangle ABC$ is isoceles, with apex B. Then by SAS, we have $\triangle ABC \cong \triangle CBA$. Therefore, $m\angle A = m\angle C$.

The proof of the converse statement is left to you as a homework exercise. □

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC (it exists and is unique by Theorem 7); the median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

Theorem 10. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*



Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and $m\angle MAB = m\angle MCB$ (by Theorem 9). Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.
Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. ☐

5. Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

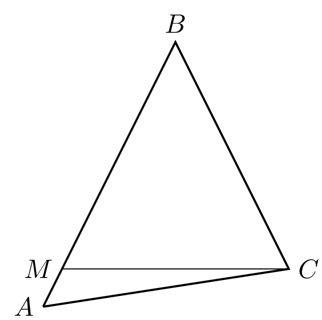
We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 9). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

Theorem 11. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.
But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 9, $m\angle A = m\angle C$ as base angles, which gives a contradiction.
Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$. On the other hand, by Problem 5, we have $m\angle BMC > m\angle A$, and by Axiom 3, we have $m\angle C = m\angle ACM + m\angle MCB > m\angle MCB$, so

$$m\angle C > m\angle MCB = m\angle BMC > m\angle A$$

so we have reached a contradiction.
Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction. Therefore, the only possibility is that $BC > AB$.



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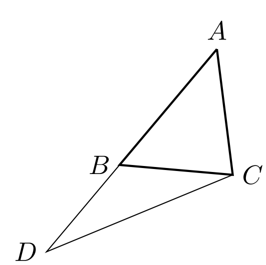
The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

Theorem 12. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 13 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*



Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 11, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) ☐

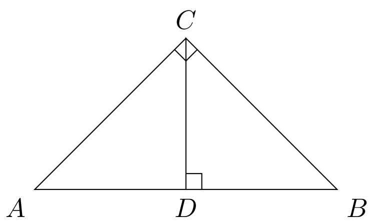
6. Homework

1. Notice that SSA and AAA are not listed as congruence rules.

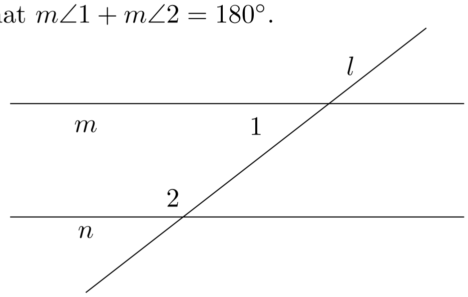
(a) Describe a pair of triangles that have two congruent sides and one congruent angle but are not congruent triangles.

(b) Describe a pair of triangles that have three congruent angles but are not congruent triangles.

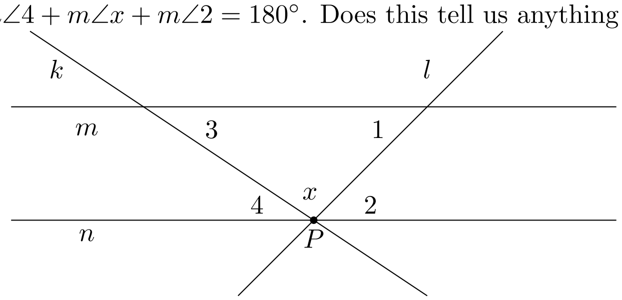
(c) In the diagram below, let $m\angle ABC = 45°$. Prove that, in $\triangle ABC$ and $\triangle BCD$, we have $\angle ABC \cong \angle BCD$, $\angle BCA \cong \angle CDB$, and $\angle CAB \cong \angle DBC$. Then notice that $\overline{BC}$ in $\triangle ABC$ is congruent to $\overline{BC}$ in $\triangle BCD$. Can we use the ASA congruence rule to deduce that $\triangle ABC \cong \triangle BCD$?



2. Consider lines l, m, and n are such that $m \parallel n$ and l intersects them both, as shown below. Prove that $m\angle 1 + m\angle 2 = 180°$.



3. Consider lines k, l, m, and n such that $m \parallel n$ and k, l, and n all intersect at P. Notice that $m\angle 4 + m\angle x + m\angle 2 = 180°$. Does this tell us anything about the sum $m\angle 1 + m\angle x + m\angle 3$?

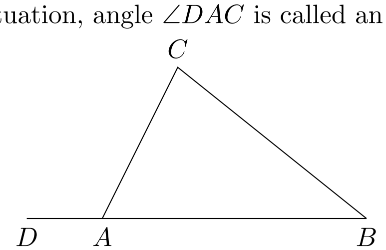


4. Prove theorem 8.

5. What is the sum of angles of a quadrialteral? a pentagon? an n-gon?

6. In a triangle $\triangle ABC$, let M be some point on the side AB (see pictrue in section *Triangle inequalitites*). Prove that then $m\angle BMC > m\angle A$.

7. Given a triangle $\triangle ABC$, let D be a point on the line AB, so that A is between D and B. In this situation, angle $\angle DAC$ is called an *external angle* of $\triangle ABC$. Prove that $m\angle DAC = m\angle B + m\angle C$.



8. Prove second part of Theorem 10: if two base angles are equal, then the triangle is isosceles.

9. Prove that the following two properties of a triangle are equivalent:

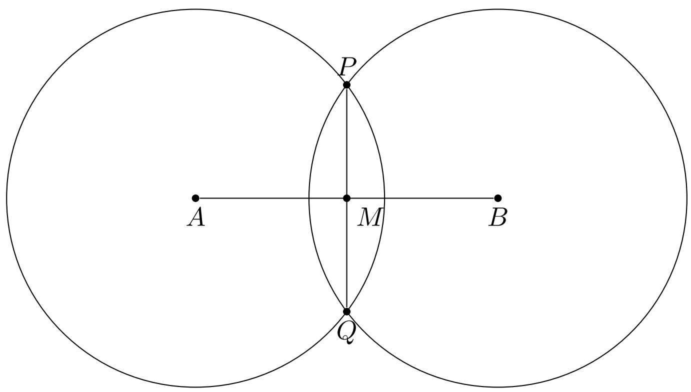
(a) All sides have the same length

(b) All angles are $60°$.

A triangle satisfying these properties is called *equilateral*.

10. The following method explains how one can find the midpoint of a segment AB using a ruler and compass:

- Choose radius r (it should be large enough) and draw circles of radius r with centers at A and B.
- Denote the intersection points of these circles by P and Q. Draw a line $\overleftrightarrow{PQ}$.
- Let M be the intersection point of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$. Then M is the midpoint of AB.



Can you justify this method, i.e., prove that so constructed point will indeed be the midpoint of AB? You can use the defining property of the circle: for a circle of radius r, the distance from any point on this circle to the center is exactly r. [Hint: find some isosceles triangles!]

11. Given $\triangle ABC$, let $\overline{CD}$ be the angle bisector of $\angle ACB$, with D on $\overline{AB}$. Suppose we wish to place a point E on $\overline{BC}$ such that $\triangle CED$ is isosceles. Prove then that we must have $AC \parallel DE$.