

## MATH 8: HANDOUT 18

### 1. POINTS AND LINES

**Axiom 1** (Lines). For any two distinct points  $A, B$ , there is a unique line containing these points (this line is usually denoted  $\overleftrightarrow{AB}$ ).

**Axiom 2** (Distance). If points  $A, B, C$  are on the same line, and  $B$  is between  $A$  and  $C$ , then  $AC = AB + BC$ .

**Axiom 3** (Angles). If point  $B$  is inside angle  $\angle AOC$ , then  $m\angle AOC = m\angle AOB + m\angle BOC$ . Also, the measure of a straight angle is equal to  $180^\circ$ .

**Axiom 4** (Euclid's Parallel Postulate). Let line  $l$  intersect lines  $m, n$  and angles  $\angle 1, \angle 2$  are as shown in the figure below (in this situation, such a pair of angles is called **alternate interior angles**). Then  $m \parallel n$  if and only if  $m\angle 1 = m\angle 2$ .

**Theorem 1.** If lines  $l, m$  intersect, then they intersect at exactly one point.

**Theorem 2** (Parallelism). If  $l \parallel m$  and  $m \parallel n$ , then  $l \parallel n$ .

**Theorem 3** (Vertical Angles). Let  $M$  be the intersection point of line segments  $\overline{AB}$  and  $\overline{CD}$ . Then  $m\angle AMC = m\angle BMD$  (such a pair of angles are called **vertical**).

**Theorem 4** (Perpendicularity). Let  $l, m$  be intersecting lines such that one of the four angles formed by their intersection is equal to  $90^\circ$ . Then the three other angles are also equal to  $90^\circ$ . (In this case, we say that lines  $l, m$  are **perpendicular** and write  $l \perp m$ .)

**Theorem 5.** Let  $l_1, l_2$  be perpendicular to  $m$ . Then  $l_1 \parallel l_2$ . Conversely, if  $l_1 \perp m$  and  $l_2 \parallel l_1$ , then  $l_2 \perp m$ .

**Theorem 6.** Given a line  $l$  and point  $P$  not on  $l$ , there exists a unique line  $m$  through  $P$  which is parallel to  $l$ .

**Theorem 7.** Given a line  $l$  and a point  $P$  not on  $l$ , there exists a unique line  $m$  through  $P$  which is perpendicular to  $l$ .

### 2. TRIANGLES AND CONGRUENCE

**Definition 1** (Triangles). A triangle consists of three points (called the vertices of the triangle) and their three corresponding line segments (the sides of the triangle). A triangle with vertices  $A, B, C$  is written  $\triangle ABC$ , and has sides  $\overline{AB}, \overline{BC}$ , and  $\overline{CA}$ , and interior angles  $\angle ABC, \angle BCA, \angle CAB$ .

**Definition 2** (Altitude, Median, Angle Bisector). In triangle  $\triangle ABC$ ,

- The altitude from  $A$  is perpendicular to  $BC$  (it exists and is unique by Theorem 7).
- The median from  $A$  bisects  $BC$  (that is, it crosses  $BC$  at a point  $D$  which is the midpoint of  $BC$ ).
- The angle bisector bisects  $\angle A$  (that is, if  $E$  is the point where the angle bisector meets  $BC$ , then  $m\angle BAE = m\angle EAC$ ).

**Definition 3** (Isosceles Triangles). A triangle is **isosceles** if two of its sides have equal length. The two sides of equal length are called **legs**; the point where the two legs meet is called the **apex** of the triangle; the other two angles are called the **base angles** of the triangle; and the third side is called the **base**.

**Definition 4** (Congruence). Congruence between two objects of the same type indicates the following:

- If two angles  $\angle ABC$  and  $\angle DEF$  have equal measure, then they are congruent angles, written  $\angle ABC \cong \angle DEF$ .
- If the distance between points  $A, B$  is the same as the distance between points  $C, D$ , then the line segments  $\overline{AB}$  and  $\overline{CD}$  are congruent line segments, written  $\overline{AB} \cong \overline{CD}$ .
- If two triangles  $\triangle ABC, \triangle DEF$  have respective sides and angles congruent, then they are congruent triangles, written  $\triangle ABC \cong \triangle DEF$ . In particular, this means  $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, \overline{CA} \cong \overline{FD}, \angle ABC \cong \angle DEF, \angle BCA \cong \angle EFD$ , and  $\angle CAB \cong \angle FDE$ .

**Definition 5** (Locus). A locus is the set of all points satisfying a certain property.

The concept of a locus is quite versatile; we will use it to describe lines and circles, but other figures can be described as well. For example, given two points  $A, B$ , the locus of points  $P$  whose sum of distances to  $A, B$  is a fixed constant (i.e.  $AP + BP = c$  for some  $c$ ) is called an *ellipse*. But that is a fun fact, and we won't dive into this type of figure in our geometry journey for now.

**Axiom 5** (SAS Congruence). *If triangles  $\triangle ABC$  and  $\triangle DEF$  have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if  $\overline{AB} \cong \overline{DE}$ ,  $\overline{BC} \cong \overline{EF}$ , and  $\angle ABC \cong \angle DEF$ , then  $\triangle ABC \cong \triangle DEF$ .*

**Axiom 6** (ASA Congruence). *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

**Axiom 7** (SSS Congruence). *If two triangles have three sides congruent, then the triangles are congruent.*

**Theorem 8** (Triangle Angle Sum). *The sum of the three interior angles of any triangle is  $180^\circ$ .*

**Theorem 9** (Isosceles Base Angles). *If  $\triangle ABC$  is isosceles, with base  $AC$ , then  $m\angle A = m\angle C$ . Conversely, if  $\triangle ABC$  has  $m\angle A = m\angle C$ , then it is isosceles, with base  $AC$ .*

**Theorem 10** (Isosceles Median, Altitude, Angle Bisector). *If  $B$  is the apex of the isosceles triangle  $ABC$ , and  $BM$  is the median, then  $BM$  is also the altitude, and is also the angle bisector, from  $B$ .*

**Theorem 11** (Greater Sides Opposite Greater Angles). *In  $\triangle ABC$ , if  $m\angle A > m\angle C$ , then we must have  $BC > AB$ .*

**Theorem 12** (Greater Angles Opposite Greater Sides). *In  $\triangle ABC$ , if  $BC > AB$ , then we must have  $m\angle A > m\angle C$ .*

**Theorem 13** (The triangle inequality). *In  $\triangle ABC$ , we have  $AB + BC > AC$ .*

**Theorem 14** (Perpendicular Bisector). *The locus of points equidistant from a pair of points  $A, B$  is a line  $l$  which is perpendicular to  $\overline{AB}$  and goes through the midpoint of  $AB$ . This line is called the *perpendicular bisector* of  $\overline{AB}$ .*

**Theorem 15** (Angle Bisector). *For an angle  $ABC$ , the locus of points inside the angle which are equidistant from the two sides  $BA, BC$  is the ray  $\overrightarrow{BD}$  which is the angle bisector of  $\angle ABC$ .*

**Theorem 16.** *In any triangle, the following results hold.*

1. *The perpendicular bisectors of three sides meet at a single point.*
2. *The three angle bisectors meet at a single point.*
3. *The three altitudes meet at a single point.*

**Note:** In original handouts, one part of this theorem was stated as Theorem 16, and other parts were given as HW problems.

### 3. QUADRILATERALS

**Definition 6** (Quadrilaterals). A figure with four vertices, four corresponding sides, and four enclosed angles is called a *quadrilateral*; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral  $ABCD$ , vertex  $A$  is opposite vertex  $C$ ). Additionally, a quadrilateral is called

- a *parallelogram*, if both pairs of opposite sides are parallel.
- a *rhombus*, if all four sides have the same length.
- a *trapezoid*, if one pair of opposite sides are parallel. (these sides are called *bases*) and the other pair is not.
- a *rectangle*, if all its angles are equal (and thus equal to  $90^\circ$ ).

**Theorem 17** (Parallelogram). *Let  $ABCD$  be a parallelogram. Then*

- $AB = DC$ ,  $AD = BC$
- $m\angle A = m\angle C$ ,  $m\angle B = m\angle D$
- *The intersection point  $M$  of diagonals  $AC$  and  $BD$  bisects each of them.*

**Theorem 18** (Parallelogram 2).

1. Let  $ABCD$  be a quadrilateral such that opposite sides are equal:  $AB = DC$ ,  $AD = BC$ . Then  $ABCD$  is a parallelogram.
2. Let  $ABCD$  be a quadrilateral such  $AB = DC$ , and  $\overline{AB} \parallel \overline{DC}$ . Then  $ABCD$  is a parallelogram.

**Theorem 19** (Rhombus). Let  $ABCD$  be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.

**Definition 7** (Midline). A midline of a triangle  $\triangle ABC$  is a segment connecting the midpoints of any two sides; a midline of a trapezoid  $ABCD$  with bases  $\overline{AB}$ ,  $\overline{CD}$  is the segment connecting the midpoints of the non-bases  $\overline{AD}$ ,  $\overline{BC}$ .

**Theorem 20** (Midline of a Triangle). If  $DE$  is the midline of  $\triangle ABC$ , then  $DE = \frac{1}{2}AC$ , and  $\overline{AD} \parallel \overline{AC}$ .

#### 4. CIRCLES

**Definition 8.**

- A circle  $\omega$  with center at  $O$  and radius  $r$  is the set of all points  $P$  such that  $OP = r$ .
- A radius is any line segment from  $O$  to a point  $A$  on  $\omega$
- A chord is any line segment between distinct points  $A$ ,  $B$  on  $\omega$
- A diameter is a chord that passes through  $O$ ,
- A tangent line is a line that intersects the circle exactly once; if the intersection point is  $A$ , the tangent is said to be the tangent through  $A$ .

**Theorem 21** (Tangent line). Let  $A$  be a point on circle  $\omega$  centered at  $O$ , and  $m$  a line through  $A$ . Then  $m$  is tangent to  $\omega$  if and only if  $m \perp \overline{OA}$ . Moreover, there is exactly one tangent to  $\omega$  at  $A$ .

**Theorem 22** (Chord perpendicular bisector). Let  $\overline{AB}$  be a chord of circle  $\omega$  with center  $O$ . Then  $O$  lies on the perpendicular bisector of  $\overline{AB}$ . Moreover, if  $C$  is on  $\overline{AB}$ , then  $C$  bisects  $\overline{AB}$  if and only if  $\overline{OC} \perp \overline{AB}$ .

**Theorem 23.** Let  $\omega_1$ ,  $\omega_2$  be circles with centers at points  $O_1$ ,  $O_2$  that intersect at points  $A$ ,  $B$ . Then  $\overline{AB} \perp \overline{O_1O_2}$ .

**Theorem 24** (Tangent circles). Let  $\omega_1$ ,  $\omega_2$  be circles that are both tangent to line  $m$  at point  $A$ . Then these two circles have only one common point,  $A$ . Such circles are called tangent.

**Theorem 25** (Inscribed angle). Let  $A$ ,  $B$ ,  $C$  be on circle  $\lambda$  with center  $O$ . Then  $\angle ACB = \frac{1}{2}\angle AOB$ . The angle  $\angle ACB$  is said to be inscribed in  $\lambda$ .

#### 5. CONSTRUCTION

**Definition 9.** Euclidean is a straightedge-compass construction game app, available on Google Play and the App Store; you can read about it at <https://www.euclidean.xyz> and [https://twitter.com/euclidean\\_app](https://twitter.com/euclidean_app)

**Definition 10.** A straightedge is a tool that can construct a line through any two points; a compass is a tool that can construct a circle centered at a given point with radius congruent to a given line segment. A construction is a process whereby one constructs a geometric object by using tools to create other objects and then constructing any desired intersection points of the objects; a straightedge-compass construction is a construction using only straightedge and compass as tools.

In general, to provide a straightedge-compass construction of a figure, you may use any previous straightedge-compass constructions you have made as tools - for example, you may construct a parallel to any line through any point, since we have proven that a straightedge-compass construction of this is possible for any line and point.

So far, we have constructed the following items with straightedge and compass, which you may use in your constructions:

- A line segment  $\overline{CD}$  congruent to  $\overline{AB}$  given the points  $A$ ,  $B$ , and  $C$ ;
- An angle at a given ray congruent to another given angle;
- A line parallel to a line  $l$  and through a point  $P$ ;

- A line perpendicular to a line  $l$  and through a point  $P$ ;
- The perpendicular bisector of a line segment  $\overline{AB}$ ;
- The center of a circle;
- The medians, altitudes, and angle bisectors of a triangle;
- A circle that passes through three given points.

#### 6. HOMEWORK

1. Try to get through as many levels of Euclidean as possible; the goal is to do the first 5 levels (up to Epsilon).
2. Prove that a diameter is the longest chord of a circle (i.e., any chord that is not a diameter has smaller length than a diameter).
3. (a) Prove that a line cannot intersect a circle at three distinct points.  
(b) Prove that two circles cannot intersect at three distinct points.
4. Suppose circles  $\lambda, \omega$  with centers  $O, P$  respectively intersect at point  $A$  such that  $m\angle OAP = x^\circ$  for some  $x < 180$ .  
(a) Prove that the tangents to  $\lambda, \omega$  at  $A$  also intersect with an angle of measure  $x^\circ$ .  
(b) Let  $B$  be the other point of intersection of  $\lambda, \omega$ ; prove that  $\angle OBP \cong \angle OAP$ .