MATH 8: HANDOUT 19

1. Numbers!

In this assignment, we start discussion of integres and divisibility. Notation:
$\mathbb{Z}$ — all integers
$\mathbb{N}$ — positive integers: $\mathbb{N} = \{1, 2, 3 \dots\}$.

Unless stated otherwise, the word "number" means "an integer number". We will not be using any other kind of numbers until further notice.

We will use without proof some basic properties of integers:

- We have operations of addition and multiplication satisfying familiar laws (commutativity, associativity, distributivity...) and order relation ($a < b$)
- Division with remainder: for any integer a and positive integer n, we can find q, r such that

$$a = qn + r, \qquad 0 \le r < n \tag{1}$$

  Moreover, q and r are uniquely determined: they are called quotient and remainder upon division of a by n.
- In any (non-empty) set of positive integers, there is a smallest number

Everything else we will derive from these properties.

Some definitions

- We write $d|a$ if d is a divisor of a, i.e., $a = dk$ for some integer k. In this situation, we also say that a is a multiple of d, or that a is divisible by d.
- d is a common divisor of m, n if $(d|m) \wedge (d|n)$;
- l is a common multiple of m, n if $(m|l) \wedge (n|l)$;
- m, n are relatively prime if they have no common divisors other than 1
- $m > 1$ is prime if m has no divisors other than 1 and itself.
- $m > 1$ is composite if it is not prime. [Note: number 1 is neither prime nor composite.]

Homework

1. Show that if n is a positive integer, then $n^2 + 8n + 17$ is not divisible by $n + 4$.

2. (a) Show that if n is even, then kn is even for any k.
(b) Show that for any integer n, $n(n + 1)$ is even.
(c) Show that for any integer n, we have $n(n + 1)(n + 2)$ is a multiple of 3.

3. Show that if $a|b$ and $b|c$, then $a|c$.

4. Show that if a, b are divisible by d, then each of the following numbers is divisible by d:
(a) $a + b$
(b) $5a + 3b$
(c) any combination of a, b, i.e. any number of the form $xa + yb$, where x, y are integers.

5. Let $a = qb + r$.
(a) Show that if a, b are both multiples of 3, then r is also a multiple of 3.
(b) Show that each common divisor of a, b is also a divisor of r.
(c) Conversely, show that if d is a common divisor of b, r, then d is also a divisor of a.

6. Is it true that any number can be written as a product of primes?

7. Show that if $p_1, \dots, p_k$ are prime, then the number $p_1 p_2 \dots p_k + 1$ is not divisible by any of p_i.
Deduce from this that there are infinitely many primes.

8. (a) Show that for any integer n, $n^{2022} - 1$ is divisible by $n - 1$. [Hint: do you remember the formula for the sum of a geometric progression?]
(b) Show that for any integer n, $n^{2021} + 1$ is divisible by $n + 1$. [Hint: write $n = -m$.]