MATH 8: HANDOUT 13
LOGIC 5: QUANTIFIERS

REMINDER: SOME LOGIC LAWS

- Given $A \implies B$ and A, we can conclude B *(Modus Ponens)*
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! It only means that **if** A is true, then so is C.]
- Given $A \wedge B$, we can conclude A (and we can also conclude B)
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$ *(Modus Tollens)*
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ *(De Morgan Law)*
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ *(Law of contrapositive)*

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

COMMON METHODS OF PROOF

Conditional proof: To prove $A \implies B$, **assume** that A is true; derive B using this assumption.

Proof by contradiction: To prove A, assume that A is **false** and derive a contradiction (i.e., something which is always false – e.g. $B \wedge \neg B$).

Combination of the above: To prove $A \implies B$, assume that A is true and that B is false and then derive a contradiction.

QUANTIFIERS

Existential quantifier: To write statements of the form "There exists an x such that...", use existential quantifier:

$$\exists x \in A : \text{ (some statement depending on } x)$$

Here A is a set of all possible values of variable x.

Example: $\exists x \in \mathbb{R} : x^2 = 5$.

Note that following the quantifier, you must have a *statement*, i.e. something that can be true or false. Usually it is some equality or inequality. You can't write there an expression which gives numerical values (for example, $\exists x \in \mathbb{R} : x^2 + 1$) — it makes no sense.

Universal quantifier: To write statements of the form "For all values of x we have...", use universal quantifier:

$$\forall x \in A : \text{ (some statement depending on } x)$$

Here A is a set of all possible values of variable x.

Example: $\forall x \in \mathbb{R} : x^2 > 0$.

LOGIC PROOFS INVOLVING QUANTIFIERS

To prove a statement $\exists x \in A : \ldots$, it suffices to give one example of x for which the statement denoted by dots is true. You need to verify that the statement is true for that value of x, but it is not necessary to explain how you found this value, nor is it necessary to find how many such values there are.

Example: to prove $\exists x \in \mathbb{R} : x^2 = 9$, take $x = 3$; then $x^2 = 9$.

To prove a statement $\forall x \in A : \ldots$, you need to give an argument which shows that for any $x \in A$, the statement denoted by dots is true. **Considering one, two, or one thousand examples is not enough!!!**

Example: to prove $\forall x \in \mathbb{R} : x^2 + 2x + 4 > 0$, we could argue as follows. Let x be an arbitrary real number. Then $x^2 + 2x + 4 = (x+1)^2 + 3$. Since a square of a real number is always non-negative, $(x+1)^2 \geq 0$, so $x^2 + 2x + 4 = (x+1)^2 + 3 \geq 0 + 3 > 0$.

Note that this argument works for any x; it uses no special properties of x except that x is a real number.

DE MORGAN LAWS FOR QUANTIFIERS

(Assuming that A is a nonempty set).

$$\neg\Big(\forall x \in A : P(x)\Big) \iff \Big(\exists x \in A : \neg P(x)\Big)$$
$$\neg\Big(\exists x \in A : P(x)\Big) \iff \Big(\forall x \in A : \neg P(x)\Big)$$

For example, negation of the statement "All flowers are white" is "There exists a flower which is not white", or in more human language, "Some flowers are not white".

HOMEWORK

0. Go through the solution of Problem 5 from Handout 10 that we discussed in class. Make sure you understand the steps involved. See if you can come up with an easier way to solve it!

1. Write the following statements using quantifiers:

(a) All birds can fly
(b) Not all birds can fly
(c) Some birds can fly
(d) All large birds can fly
(e) Only large birds can fly
(f) No large bird can fly

You can use letter B for the set of all birds, and notation $F(x)$ for statement "x can fly" and $L(x)$ for "x is large".

2. Write the following statements using logic operations and quantifiers:
(a) All mathematicians love music
(b) Some mathematicians don't like music
(c) No one but a mathematician likes music
(d) No one would go to John's party unless he loves music or is a mathematician
Please use the following notation:
P – set of all people

$M(x)$ — x is a mathematician
$L(x)$ — x loves music
$J(x)$ — x goes to John's party

3. Write each of the following statements using only quantifiers, arithmetic operations, equalities and inequalities. In all problems, letters x, y, z stand for a variables that takes real values, and letters $m, n, k, \ldots$ stand for variables that take integer values.

(a) Equation $x^2 + x - 1$ has a solution
(b) Inequality $y^3 + 3y + 1 < 0$ has a solution
(c) Inequality $y^3 + 3y + 1 < 0$ has a positive real solution
(d) Number 100 is even.
(e) Number 100 is odd
(f) For any integer number, if it is even, then its square is also even.

4. For each of the statements of the previous problem, try to determine if it is true. If it is, give a proof. If not, disprove it (i.e., prove its negation).

***5.** Consider the following arguments:

(a) No homework is fun.
Some reading is homework.
Therefore, some reading is not fun.

(b) All informative things are useful.
Some websites are not useful.
Some websites are not informative.

For each of them,

(1) write it in symbolic form, using quantifiers. Use set A for set of all human activities (which includes reading, writing, homework, etc) and notation $F(x)$ for "x is fun", $H(x)$ for "x is homework", etc) and
(2) prove it, using the methods of proof discussed in class.

FYI: these are examples of two of Aristotle's syllogisms, namely *Ferio* and *Baroco*. For more information, google Aristotle and syllogism.

6. There is a way of shuffling a deck of cards known as the *52-card pickup*. To perform this shuffle, one takes a deck of 52 cards, throws all the cards onto a table, and then picks them back up, one at a time and randomly, until one has the entire deck back in one's hand. What is the probability that shuffling the deck like this will return the cards in the original order?

7. A shower curtain hangs from a metal bar via 10 hooks. If I unhook some of these hooks, then as long as there is at least one hook still hooked, the curtain will not completely fall down. In how many ways can I unhook a selection of the curtain's hooks so that the curtain does not completely fall down?

8. Mr. Mime and Ninetails have an $m \times n$ chocolate bar, which can be broken along any of its edges to produce smaller rectangular bars. They decide to play a game, where they take turns splitting the chocolate along one edge at a time: Mr. Mime will go first, splitting the bar along one edge, and then Ninetails will go second, splitting one of the subsequent pieces along one of its edges, and then Mr. Mime will split one of the pieces after that, etc. Once the bar is reduced completely to 1×1 squares, then the last player to have made a move is the winner and gets to eat all the chocolate. Note that they may only split one piece at a time!

Is there a winning strategy for this game for either player?