# MATH 8: HANDOUT 17
# EUCLIDEAN GEOMETRY 4: QUADRILATERALS. MIDLINE OF A TRIANGLE.

## 9. Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex $A$ is opposite vertex $C$). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex $A$ is opposite vertex $C$, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

**Definition.** A quadrilateral is called

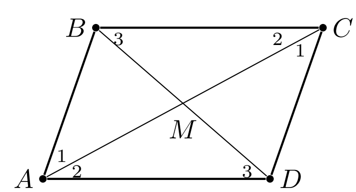
- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

**Theorem 14.** *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point $M$ of diagonals $AC$ and $BD$ bisects each of them.*

*Proof.* Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

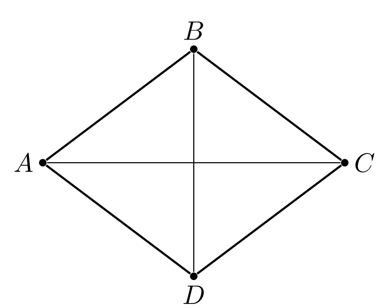



Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. □

**Theorem 15.** *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*

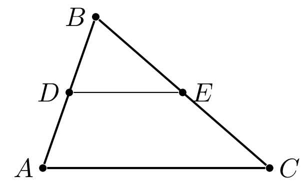
Proof is left to you as a homework exercise.

**Theorem 16.** *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*




*Proof.* Since the opposite sides of a rhombus are equal, it follows from Theorem 15 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let $M$ be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and $BM$ is a median, by Theorem 13 i, it is also the altitude. □

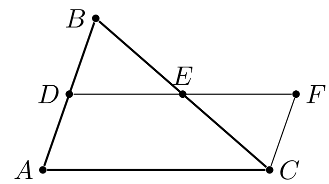
## 10. MIDLINE OF A TRIANGLE




**Definition.** A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two sides.

**Theorem 17.** *If $DE$ is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

*Proof.* Continue line $DE$ and mark on it point $F$ such that $DE = EF$.




1. $\triangle DEB \cong \triangle FEC$ by SAS: $DE = EF$, $BE = EC$, $\angle BED \cong \angle CEF$.
2. $ADFC$ is a parallelogram: First, we can see that since $\triangle DEB \cong \triangle FEC$, then $\angle BDE \cong \angle CFE$, and since they are alternate interior angles, $AD \parallel FC$. Also, from the same congruency, $FC = BD$, but $BD = AD$ since $D$ is a midpoint. Then, $FC = DA$. So we have $FC = DA$ and $FC \parallel DA$, and therefore $ADFC$ is a parallelogram.
3. That gives us the second part of the theorem: $DE \parallel AC$. Also, since $ADFC$ is a parallelogram, $AC = DF = 2 \cdot DE$, and from here we get $DE = \frac{1}{2}AC$.

□

### HOMEWORK

**Note that you may use all results that are presented in the previous sections.** This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

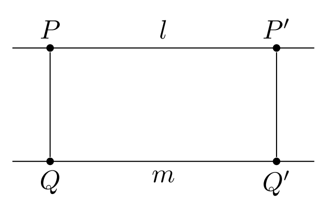
1. (Slant lines and perpendiculars) Let $P$ be a point not on line $l$, and let $Q \in l$ be such that $PQ \perp l$. Prove that then, for any other point $R$ on line $l$, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.

   **Note**: you can not use the Pythagorean theorem for this, as we haven't yet proved it! Instead, use Theorem 11.
2. Prove that in any triangle, the three angle bisectors intersect at a single point (compare with the similar fact about perpendicular bisectors – Problem 8 from Handout 14)
3. **[We did it in class]**(Parallelogram) Who doesn't love parallelograms?
   (a) Prove Theorem 15.
   (b) Prove that if in a quadrilateral $ABCD$ we have $AD = BC$, and $\overline{AD} \parallel \overline{BC}$, then $ABCD$ is a parallelogram.
4. Prove that in a parallelogram, sum of two adjacent angles is equal to $180°$:

$$m\angle A + m\angle B = m\angle B + m\angle C = \cdots = 180°$$

5. (Rectangle) A quadrilateral is called rectangle if all angles have measure $90°$.
   (a) Show that each rectangle is a parallelogram.
   (b) Show that opposite sides of a rectangle are congruent.
   (c) Prove that the diagonals of a rectangle are congruent.
   (d) Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.
6. (Distance between parallel lines)

   Let $l, m$ be two parallel lines. Let $P \in l$, $Q \in m$ be two points such that $\overleftrightarrow{PQ} \perp l$ (by Theorem 6, this implies that $\overleftrightarrow{PQ} \perp m$).
   Show that then, for any other segment $P'Q'$, with $P' \in l$, $Q' \in m$ and $\overleftrightarrow{P'Q'} \perp l$, we have $PQ = P'Q'$. (This common distance is called the distance between $l$, $m$.)



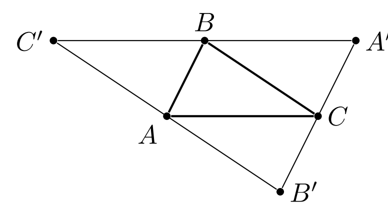
**7.** **[We did it in class]** The following statements about a parallelogram can be used as its definition, i.e. you can prove any of them from any other. Can you show how?

We have done some of the proofs already. Establish which other statements need to be proven to show the equivalence of all of these statements, and try to prove them. For example, Theorem 15 proves (b) $\Rightarrow$ (a), and Theorem 14 proves (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), and (a) $\Rightarrow$ (d); (e) $\Rightarrow$ (a) is proven in Problem 4b.

(a) Opposite sides are parallel.
(b) Opposite sides are congruent.
(c) Opposite angles are congruent.
(d) Diagonals bisect each other.
(e) One pair of opposite sides is parallel and congruent.

**8.** Show that if we mark midpoints of each of the three sides of a triangle, and connect these points, the resulting segments will divide the original triangle into four triangles, all congruent to each other.

**9.** (Altitudes intersect at single point)
The goal of this problem is to prove that three altitudes of a triangle intersect at a single point. Given a triangle $\triangle ABC$, draw through each vertex a line parallel to the opposite side. Denote the intersection points of these lines by $A', B', C'$ as shown in the figure.




(a) Prove that $A'B = AC$ (hint: use parallelograms!)
(b) Show that $B$ is the midpoint of $A'C'$, and similarly for other two vertices.
(c) Show that altitudes of $\triangle ABC$ are exactly the perpendicular bisectors of sides of $\triangle A'B'c'$.
(d) Prove that the three altitudes of $\triangle ABC$ intersect at a single point.