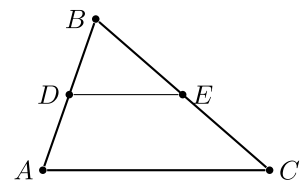
MATH 8: HANDOUT 18
EUCLIDEAN GEOMETRY – 5: MIDLINE. CIRCLES. INSCRIBED ANGLES.

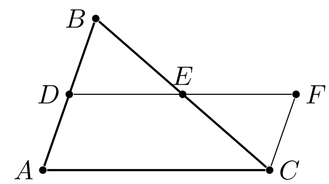
10. Midline of a triangle and trapezoid



Definition. A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two sides.

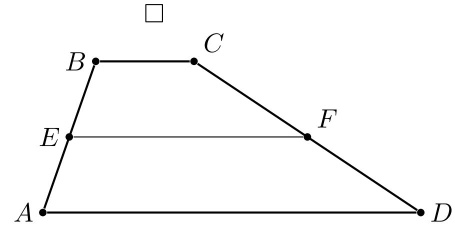
Theorem 17. *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

Proof. Continue line DE and mark on it point F such that $DE = EF$.

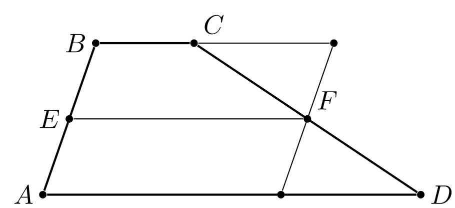


1. $\triangle DEB \cong \triangle FEC$ by SAS: $DE = EF$, $BE = EC$, $\angle BED \cong \angle CEF$.
2. $ADFC$ is a parallelogram: First, we can see that since $\triangle DEB \cong \triangle FEC$, then $\angle BDE \cong \angle CFE$, and since they are alternate interior angles, $AD \parallel FC$. Also, from the same congruency, $FC = BD$, but $BD = AD$ since D is a midpoint. Then, $FC = DA$. So we have $FC = DA$ and $FC \parallel DA$, and therefore $ADFC$ is a parallelogram.
3. That gives us the second part of the theorem: $DE \parallel AC$. Also, since $ADFC$ is a parallelogram, $AC = DF = 2 \cdot DE$, and from here we get $DE = \frac{1}{2}AC$.

□



Theorem 18 (Trapezoid midline)**.** *Let $ABCD$ be a trapezoid, with bases AD and BC, and let E, F be midpoints of sides AB, CD respectively. Then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.*



Idea of the proof: draw through point F a line parallel to AB, as shown in the figure. Prove that this gives a parallelogram, in which points E, F are midpoints of opposite sides.

11. Constructions with straightedge and compass

Large part of classical geometry are geometric constructions: can we construct a figure with given properties? Traditionally, such constructions are done using straight-edge and compass: the straight-edge tool constructs lines and the compass tool constructs circles. More precisely, it means that we allow the following basic operations:

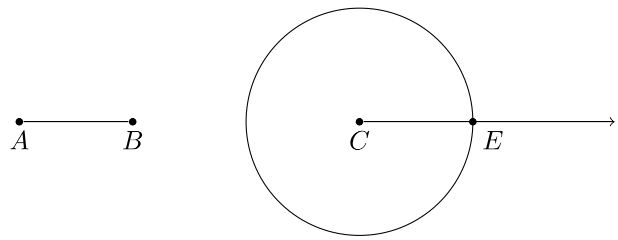
- Draw (construct) a line through two given or previously constructed distinct points. (Recall that by axiom 1, such a line is unique).
- Draw (construct) a circle with center at previously constructed point O and with radius equal to distance between two previously constructed points B, C
- Construct the intersections point(s) of two previously constructed lines, circles, or a circle and a line

All other constructions (e.g., draw a line parallel to a given one) must be done using these elementary constructions only!

Constructions of this form have been famous since mathematics in ancient Greece. Here are some examples of constructions:

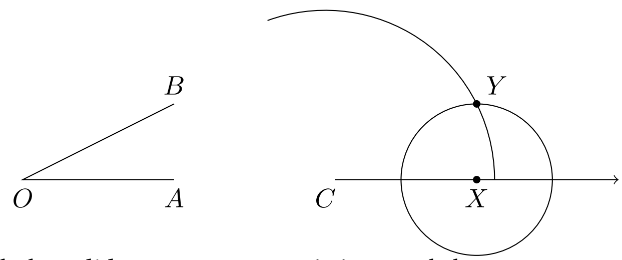
Example 1. Given any line segment $\overline{AB}$ and ray $\overrightarrow{CD}$, one can construct a point E on $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

Construction. Construct a circle centered at C with radius AB. Then this circle will intersect $\overrightarrow{CD}$ at the desired point E. □



Example 2. Given angle $\angle AOB$ and ray $\overrightarrow{CD}$, one can construct an angle around $\overrightarrow{CD}$ that is congruent to $\angle AOB$.

Construction. First construct point X on $\overrightarrow{CD}$ such that $CX \cong OA$. Then, construct a circle of radius OB centered at C and a circle of radius AB centered at X. Let Y be the intersection of these circles; then $\triangle XCY \cong \triangle AOB$ by SSS and hence $\angle XCY \cong \angle AOB$. □



A great tool to learn these constructions is an app called *Euclidea*. You can use it in a web browser at `http://euclidea.xyz`, or install it on your phone or tablet (it is available both for iOS and Android).

Note: Euclidea starts with a slightly more restrictive set of tools. Namely, it only allows drawing circles with a given center and passing through a given point; thus, you can not use another segment as radius.

12. Circles

Definition. A circle with center O and radius $r > 0$ is the set of all points P in the plane such that $OP = r$.

Traditionally, one denotes circles by Greek letters: $\lambda, \omega \dots$.
Given a circle λ with center O,

- A radius is any line segment from O to a point A on λ,
- A chord is any line segment between distinct points A, B on λ,
- A diameter is a chord that passes through O,

Recall that by Theorem 16, if O is equidistant from points A, B, then O must lie on the perpendicualr bisector of AB. We can restate this result as follows.

Theorem 19. *If AB is a chord of circle λ, then the center O of this circle lies on the perpendicular bisector of AB.*

13. Relative positions of lines and circles

Theorem 20. *Let λ be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l, i.e. the length of the perpendicular OP from O to l. Then:*

- *If $d > r$, then λ and l do not intersect.*
- *If $d = r$, then λ intersects l at exactly one point P, the base of the perpendicular from O to l. In this case, we say that l is tangent to λ at P.*
- *If $d < r$, then λ intersects l at two distinct points.*

Proof. First two parts easily follow from Theorem 14: slant line is longer than the perpendicular.

For the last part, it is easy to show that λ can not intersect l at more than 2 points (see problem 1 of previous homework). Proving that it does intersect l at two points is very hard and requires deep results about real numbers. This proof will not be given here. □

Note that it follows from the definition that a tangent line is perpendicular to the radius OP at point of tangency. Converse is also true.

Theorem 21. *Let λ be a circle with center O, and let l be a line through a point A on λ. Then l is tangent to λ if and only if $l \perp \overleftrightarrow{OA}$*

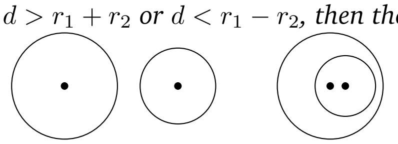
Proof. By definition, if l is the tangent line to λ, then it has only one common point with λ, and this point is the base of the perpendicular from O to l; thus, OA is the perpendicular to l.

Conversely, if $OA \perp l$, it means that the distance from l to O is equal to the radius (both are given by OA), so l is tangent to λ. $\square$

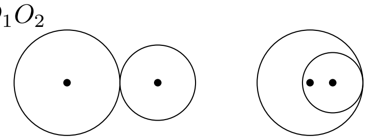
Similar results hold for relative position of a pair of circles. We will only give part of the statement.

Theorem 22. *Let λ_1, λ_2 be two circles, with centers O_1, O_2 and radiuses r_1, r_2 respectively; assume that $r_1 \geq r_2$. Let $d = O_1O_2$ be the distance between the centers of the two circles.*

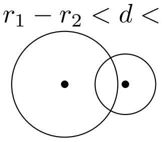
- *If $d > r_1 + r_2$ or $d < r_1 - r_2$, then these two circles do not intersect.*



- *If $d = r_1 + r_2$ or $d = r_1 - r_2$ then these two circles have a unique common point, which lies on the line O_1O_2*



- *If $r_1 - r_2 < d < r_1 + r_2$, then the two circles intersect at exactly two points.*



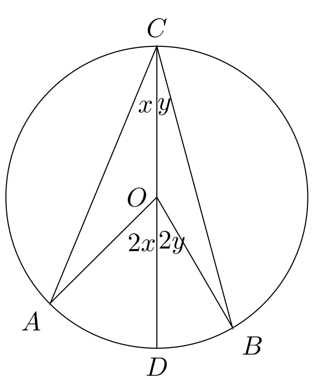
We skip the proof.

Definition. Two circles are called tangent if they intersect at exactly one point.

14. Arcs and Angles

Consider a circle λ with center O, and an angle formed by two rays from O. Then these two rays intersect the circle at points A, B, and the portion of the circle contained inside this angle is called the arc subtended by $\angle AOB$. We will sometimes use the notation $\overset{\frown}{AB}$. We define the measure of the arc as the measure of the corresponding central angle: $\overset{\frown}{AB} = m\angle AOB$.

Theorem 23. *Let A, B, C be on circle λ with center O. Then $m\angle ACB = \frac{1}{2}\overset{\frown}{AB}$. The angle $\angle ACB$ is said to be inscribed in λ.*



Proof. There are actually a few cases to consider here, since C may be positioned such that O is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.

Case 1. Draw diameter $\overline{CD}$. Let $x = m\angle ACD$, $y = m\angle BCD$, so that $m\angle ACB = x + y$.

Since $\overline{OC}$ is a radius of λ, we have that $\triangle AOC$ is isosceles triangle, thus $m\angle A = x$. Therefore, $m\angle AOD = 2x$, as it is the external angle of $\triangle AOC$. Similarly, $m\angle BOD = 2y$. Thus, $\overset{\frown}{AB} = \overset{\frown}{AD} + \overset{\frown}{DB} = 2x + 2y$. $\square$

This theorem has a converse, which essentially says that **all** points C forming a given angle $\angle ACB$ with gven points A, B must lie on a circle containing points A, B. Exact statement is given in the homework (see problem **??**).

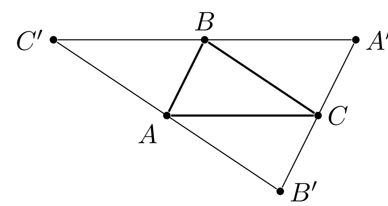
As an immediate corollary, we get the following result:

Theorem 24. *Let λ be a circle with diameter AB. Then for any point C on this circle other than A, B, the angle $\angle ACB$ is the right angle. Conversely, if a point C is such that $\angle ACB$ is the right angle, then C must lie on the circle λ.*

HOMEWORK

1. Show that if we mark midpoints of each of the three sides of a triangle, and connect these points, the resulting segments will divide the original triangle into four triangles, all congruent to each other.

2. (Altitudes intersect at single point)
The goal of this problem is to prove that three altitudes of a triangle intersect at a single point. Given a triangle $\triangle ABC$, draw through each vertex a line parallel to the opposite side. Denote the intersection points of these lines by A', B', C' as shown in the figure.



(a) Prove that $A'B = AC$ (hint: use parallelograms!)
(b) Show that B is the midpoint of $A'C'$, and similarly for other two vertices.
(c) Show that altitudes of $\triangle ABC$ are exactly the perpendicular bisectors of sides of $\triangle A'B'C'$.
(d) Prove that the three altitudes of $\triangle ABC$ intersect at a single point.

3. Without using Theorem 20, prove that a circle can not have more than two intersections with a line. [Hint: assume it has three intersection points, and use Theorem 19 to get a contradiction.]

4. Prove that given three points A, B, C not on the same line, there is a unique circle passing through these points. This circle is called the circumscribed circle of $\triangle ABC$. Explain how to construct this circle using ruler and compass.

5. Show that if a circle ω is tangent to both sides of the angle $\angle ABC$, then the center of that circle must lie on the angle bisector. [Hint: this center is equidistant from the two sides of the circle.] Show that conversely, given a point O on the angle bisector, there exists a circle with center at this point which is tangent to both sides fo the angle.

6. Use the previous problem to show that for any triangle, there is a unique circle that is tangent to all three sides (inscribed circle).

7. Given a circle λ with center A and a point B outside this circle, construct the tangent line l from B to λ using straightedge and compass. How many solutions does this problem have?

[Hint: let P be the tangency point (which we haven't contructed yet). Then by Theorem 21, $\angle APB$ is a right angle. Thus, by Theorem 24, it must lie on a circle with diameter OP]

8. Complete levels α, β in Euclidea.