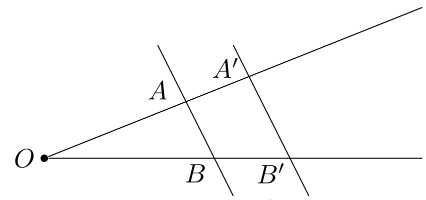
MATH 8: HANDOUT 19
EUCLIDEAN GEOMETRY 6: SIMILAR TRIANGLES.

15. Thales Theorem



Theorem 25 (Thales Theorem)**.** *Let points A', B' be on the sides of angle $\angle AOB$ as shown in the picture. Then lines AB and $A'B'$ are parallel if and only if*

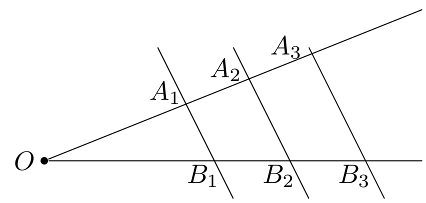
$$\frac{OA}{OB} = \frac{OA'}{OB'}$$

In this case, we also have $\frac{OA}{OB} = \frac{AA'}{BB'}$

We have already seen and proved a special case of this theorem when discussing the midline of a triangle.

The proof of this theorem is unexpectedly hard. In the case when $\frac{OA}{OA'}$ is a rational number, one can use arguments similar to those we did when talking about midline. The case of irrational numbers is harder yet. We skip the proof for now; it will be discussed in Math 9.

As an immediate corollary of this theorem, we get the following result.



Theorem 26. *Let points $A_1, \dots, A_n$ and $B_1, \dots B_n$ on the sides of an angle be chosen so that $A_1A_2 = A_2A_3 = \dots = A_{n-1}A_n$, and lines A_1B_1, A_2B_2, ... are parallel. Then $B_1B_2 = B_2B_3 = \dots = B_{n-1}B_n$.*

Proof of this theorem is left to you as exercise.

16. Similar triangles

Definition. Two triangles $\triangle ABC$, $\triangle A'B'C'$ are called *similar* if

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \angle C \cong \angle C'$$

and the corresponding sides are proportional, i.e.

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

The common ratio $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ is sometimes called the similarity coefficient.

There are some similarity tests:

Theorem 27 (AAA similarity test)**.** *If the corresponding angles of triangles $\triangle ABC$, $\triangle A'B'C'$ are equal:*

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \angle C \cong \angle C'$$

then the triangles are similar.

Theorem 28 (SSS similarity test)**.** *If the corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

then the triangles are similar.

Theorem 29 (SAS similarity test)**.** *If two pairs of corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

and $\angle A \cong \angle A'$ then the triangles are similar.

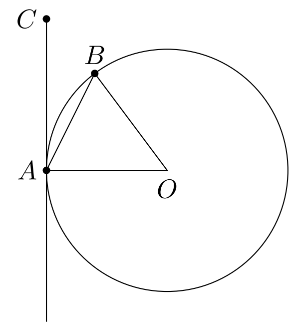
Proofs of all of these tests can be obtained from Thales theorem.

Homework

1. (Angle Theorems) Let's study Inscribed Angle Theorem (Theorem 23 from Handout 16) in a bit more detail!

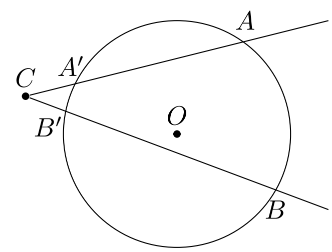
(a) Prove the converse of this theorem: namely, if λ is a circle centered at O and A, B, are on λ, and there is a point C such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then C lies on λ. [Hint: let C' be the point where line AC intersects λ. Show that then, $m\angle ACB = m\angle AC'B$, and show that this implies $C = C'$.]

(b) Let A, B be on circle λ centered at O and m the tangent to λ at A, as shown on the right. Let C be on m such that C is on the same side of $\overleftrightarrow{OA}$ as B. Prove that $m\angle BAC = \frac{1}{2}m\angle BOA$. [Hint: extend $\overline{OA}$ to intersect λ at point D so that $\overline{AD}$ is a diameter of λ. What arc does $\angle DAB$ subtend?]



2. Here is a modification of Inscribed Angle Theorem.

Consider a circle λ and an angle whose vertex C is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, intersection of the angle with the circle defines two arcs: $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$.
Prove that in this case, $m\angle C = \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{A'B'})$.
[Hint: draw line AB' and find first the angle $\angle AB'B$. Then notice that this angle is an exterior angle of $\triangle ACB'$.]



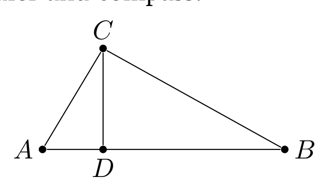
3. Can you suggest and prove an analog of the previous problem, but when the point C is inside the circle (you will need to replace an angle by two intersecting lines, forming a pair of vertical angles)?

4. Prove Theorem 26 (using Thales Theorem). Hint: let $k = \frac{OB_1}{OA_1}$; show that then $B_iB_{i+1} = kA_iA_{i+1}$.

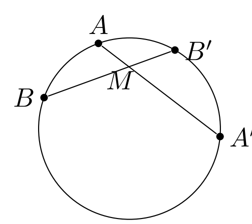
5. Using Theorem 26, describe how one can divide a given segment into 5 equal parts using ruler and compass.

6. Given segments of length a, b, c, construct a segment of length $\frac{ab}{c}$ using ruler and compass.

7. Let ABC be a right triangle, $\angle C = 90°$, and let CD be the altitude. Prove that triangles $\triangle ACD$, $\triangle CBD$ are similar. Deduce from this that $CD^2 = AD \cdot DB$.



8. Let M be a point inside a circle and let AA', BB' be two chords through M. Show that then $AM \cdot MA' = BM \cdot MB'$. [Hint: use inscribed angle theorem to show that triangles $\triangle AMB$, $\triangle B'MA'$ are similar.]



9. Let AA', BB' be altitudes in the acute triangle $\triangle ABC$.

(a) Show that points A', B' are on a circle with diameter AB.
(b) Show that $\angle AA'B' = \angle ABB'$, $\angle A'B'B = \angle A'AB$
(c) Show that triangle $\triangle ABC$ is similar to triangle $\triangle A'B'C$.