September 26, 2021

Baseline revue test. Algebra.

1. Open brackets and expand the following expressions
   a. $(a+b)^2 =$
   b. $(a-b)^2 =$
   c. $(a+b)^3 =$
   d. $(a-b)^3 =$
2. Factor the following expressions
   a. $a^2 - b^2 =$
   b. $a^2 + b^2 =$
   c. $a^3 - b^3 =$
   d. $a^3 + b^3 =$
   e. $1 + a + a^2 + a^3 =$
3. For a quadratic equation $ax^2 + bx + c = 0$ the roots are,
   $x_{1,2} =$
   and they have the following properties,
   $x_1 + x_2 =$
   $x_1 \cdot x_2 =$
4. Open parentheses and expand the following expression,

   $(a+b)^{10} =$

5. What is the number of permutations of n objects?
6. How many ways is there to select k objects out of n if,
   a. order does matter?
   b. order does not matter?
7. Write the formula for a binomial coefficient

   $$C_n^k \equiv {}_nC_k \equiv \binom{n}{k} =$$

   and explain its relation to combinatorics and certain counting problems.

Solutions to the baseline revue test. Algebra.

1. Open brackets and expand the following expressions
   a. $(a+b)^2 = a^2 + 2ab + b^2$
   b. $(a-b)^2 = a^2 - 2ab + b^2$
   c. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
   d. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
2. Factor the following expressions
   a. $a^2 - b^2 = (a-b)(a+b)$
   b. $a^2 + b^2 = \ldots$
   c. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
   d. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
   e. $1 + a + a^2 + a^3 = (1+a)(1+a^2) = \frac{1-a^4}{1-a}$

   The last example was a particular case of a geometric progression, whose sum is one of the most important expressions in algebra,

$$1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$$

   A simple heuristic way to prove this result is to multiply both sides with $(1-a)$ and then open the brackets,

$$(1 + a + a^2 + \cdots + a^n)(1-a) = 1 - a + a - a^2 + a^2 - \cdots - a^n + a^n - a^{n+1} = 1 - a^{n+1}$$

3. For a quadratic equation $ax^2 + bx + c = 0$ the roots are,
   $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   and they have the following properties,
   $x_1 + x_2 = -\frac{b}{a}$
   $x_1 \cdot x_2 = \frac{c}{a}$

Although this can be checked by direct substitution of the formula for $x_{1,2}$, the easiest way to see this is by rewriting the equation in the reduced form and identifying it with the product of the two brackets,

$$ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Leftrightarrow (x - x_1)(x - x_2) = 0 \Leftrightarrow x^2 - (x_1 + x_2)x + x_1x_2 = 0$$

This is an example of the polynomial factorization, which we will be studying in significant detail this year.

4. Open parentheses and expand the following expression,

$$(a+b)^n = a^n + na^{n-1}b + {}_nC_k a^{n-2}b^2 + {}_nC_k a^{n-k}b^k + \cdots + {}_nC_1 a\, b^{n-1} + b^n$$

This is the Newton's binomial formula, and we will be reviewing and re-deriving it later this year using the mathematical induction.

5. What is the number of permutations of n objects? Answer: $n!$

This is the number of ways that n different objects (or subjects) can be placed into n different places.

Examples:

- How many ways is there to sit n people in a movie theater with n numbered chairs?
- How many ways is there to hand out n different books to n students?
- How many ways is there to place n numbered billiard balls into n numbered spots?

There is n ways to select a place for the first object (subject), for each of these n choices there is $n-1$ choice to place the second one, so there are $n(n-1)$ in total different choices to fill the first two spots, and so on. Hence, there are $n! = n(n-1)(n-2) \dots \cdot 2 \cdot 1$.

6. How many ways is there to select k objects out of n if,
   a. order does matter? Answer: ${}_nP_k = \frac{n!}{(n-k)!}$
   b. order does not matter? Answer: ${}_nC_k = \frac{n!}{k!(n-k)!}$

Math 9 placement test 2021

1. Open the parentheses and expand the following expressions
    a. $(a+3b)^2 =$
    b. $(a-b)^3 =$
2. Solve the inequality

$$x + \frac{1}{x} > 4.25$$

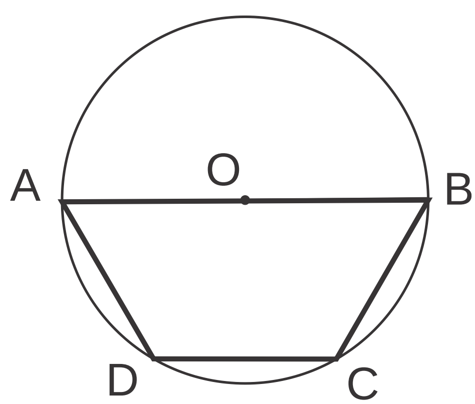
3. Factor the following expressions
    a. $a^2 - b^2 =$
    b. $a^3 - b^3 =$
    c. $a^3 + b^3 =$
    d. $1 + a + a^2 + a^3 =$
4. Alice, Bob, Charlie, and Diana are buying ice cream. The store has 15 different flavors of ice cream. How many ways there are for them to choose if each kid chooses one ice cream flavor? How will the answer change if in addition, we require that no two of them choose the same flavor?
5. Find the coefficient of x^2 in the polynomial $(x+2)^8$
6. How many divisors does the number 360 have?
7. What is the last digit of of 7^{2021}.
8. Given angle $\angle AOB$, construct the angle bisector using straightedge and compass. (You can do your construction online here: https://www.geogebra.org/geometry/nrn7t5nr)

Math 9 placement test 2020

1. Open brackets and expand the following expressions
    a. $(a+b)^2 =$
    b. $(a-b)^3 =$
2. Factor the following expressions
    a. $a^2 - b^2 =$
    b. $a^3 - b^3 =$
    c. $a^3 + b^3 =$
    d. $1 + a + a^2 + a^3 =$
3. Find the remainder of 2020^{2020} upon division by 3.
4. Solve the equation

$$x + \frac{1}{x} = 4\frac{1}{4}$$

5. Eight teams have reached the quarter-finals of the soccer World Cup.
    a. How many ways are there for these teams to be paired to play the quarter-final games?
    b. How many different outcomes of which team wins which medal (gold, silver, bronze) are possible?
6. Trapezoid $ABCD$ is inscribed in a circle of radius r as shown in the Figure. The longer base of the trapezoid is equal to diameter, $|AB| = 2r$, while the shorter bas equals r. Find the area of the trapezoid $ABCD$.

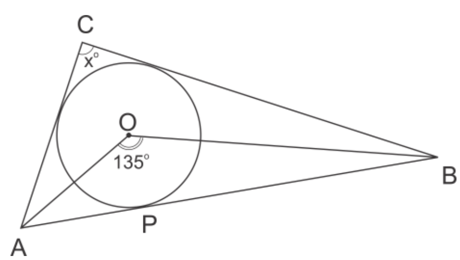


Math 9 placement test 2019

1. Factor the following expressions
   a. $a^2 - b^2 =$
   b. $a^3 - b^3 =$
   c. $a^3 + b^3 =$
   d. $1 + a + a^2 + a^3 =$
2. Find the coefficient of x^7 in the polynomial $(1 + 2x)^9$.
3. Let x_1, x_2 be roots of the equation $x^2 = x + 1$. Find $\frac{1}{x_1} + \frac{1}{x_2}$.
4. Find the remainder of 2^{2019} upon division by 7.
5. Let $ABCD$ be a trapezoid, with bases AD and BC, and let E, F be midpoints of sides AB, CD respectively. If $BC = 2$ cm, $AD = 6$ cm, then what is EF? Can you prove your answer?

Math 9 placement test 2018

6. Open brackets and expand the following expressions
   a. $(a + b)^2 =$
   b. $(a - b)^3 =$
7. Factor the following expressions
   a. $a^2 - b^2 =$
   b. $a^3 - b^3 =$
   c. $a^3 + b^3 =$
   d. $1 + a + a^2 + a^3 =$
8. Solve the following inequality. Write your answer as a set of possible values for x.
$$\frac{(x+2)^2(x-7)}{x+3} \le 0$$
9. Find the remainder of 2^{2019} upon division by 7.
10. Let x_1, x_2 be roots of the equation $x^2 = x + 1$. Find $\frac{1}{x_1} + \frac{1}{x_2}$.
11. Find the remainder upon division of 2^{2019} by 7.
12. O is the center of the inscribed circle in triangle ABC. The angle AOB is 135 degrees. Find the angle ACB.



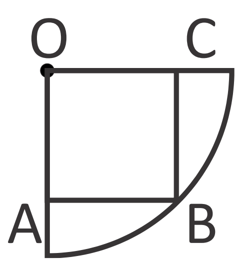
Math 9 placement test 2017

1. Open brackets and expand the following expressions
    a. $(a+b)^2 =$
    b. $(a-b)^3 =$
2. Factor the following expressions
    a. $a^2 - b^2 =$
    b. $a^3 - b^3 =$
    c. $a^3 + b^3 =$
    d. $1 + a + a^2 + a^3 =$
3. Find the remainder of 3^{2017} upon division by 4.
4. Solve the equation

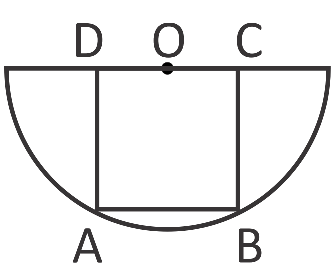
$$x + \frac{1}{x} = 7\frac{1}{7}$$

5. Eight teams have reached the quarter-finals of the soccer World Cup.
    a. How many ways are there for these teams to be paired to play the quarter-final games?
    b. How many different outcomes of which team wins which medal (gold, silver, bronze) are possible?
6. Find the area of a square inscribed in

    a. a quarter circle of radius r, as shown in the Figure below,



    b. a semicircle circle of radius r as shown in the Figure below.



Math 9 placement test 2016

1. Open brackets and expand the following expressions
   a. $(a+b)^2 =$
   b. $(a-b)^3 =$
2. Factor the following expressions
   a. $a^2 - b^2 =$
   b. $a^3 - b^3 =$
   c. $a^3 + b^3 =$
   d. $1 + a + a^2 + a^3 =$
3. Find the remainder of 3^{2016} upon division by 5.
4. Solve the equation

$$\frac{x^2+1}{x} - \frac{2x}{x^2+1} = 1$$

5. Eight teams have reached the quarter-finals of the soccer World Cup.
   a. How many ways are there for these teams to be paired to play the quarter-final games?
   b. How many different outcomes of which team wins which medal (gold, silver, bronze) are possible?
6. Four equal segments are cut off a circle of radius r so that a square is obtained. Find the area of each of these segments.

Math 8-9 placement test 2015

1. How many ways are there to choose a team captain and 6 team members out of 15 candidates?
2. If x_1, x_2 are roots of the square equation $x^2 + 2x - 7$, what is $x_1 x_2$? $\frac{1}{x_1} + \frac{1}{x_2}$?
3. Simplify the following expression

$$\frac{(a^2 - b^2)^3}{(a - b)(a + b)^2}$$

4. How many "words" can you form by permuting the letters of the word "letter"? (A "word" is any combination of letters, not necessarily meaningful)
5. Points $A = (0, 0)$, $B = (2, 0)$, and C on the coordinate plane form an equilateral triangle. What are the coordinates of point C?
6. Factor $a^4 - b^4$.
7. Corners of a square with the side a are cut off so that a regular octagon is obtained. Find the area of this octagon.
8. Solve the inequality

$$\frac{x + 5}{x^2 - 2x - 3} > 0$$

9. If we write the polynomial $(x + 2)^{10}$ in the usual form $x^{10} + a_1 x^9 + a_2 x^8 + \ldots$, what would be the coefficient of x^6?
10. Find all integer numbers which give remainder 2 upon division by 7 and remainder 5 upon division by 13.
11. Given triangle ABC, explain how to construct (using ruler and compass) a point which is at equal distance from points A, B, and C.

Math 8-9 placement test 2015: solutions to selected problems

(1) $15 \cdot C_{14}^{6} = 15 \cdot \frac{14!}{6!\cdot 8!} = \frac{15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$

$= 15\cdot 7\cdot 13\cdot 11\cdot 3 = \boxed{45045}$

(2) $x_1+x_2=-2$, $\quad x_1\cdot x_2=-7$

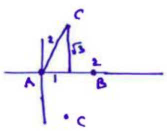
$\frac{x_1+x_2}{x_1\cdot x_2} = \frac{1}{x_1}+\frac{1}{x_2} = \frac{-2}{-7} = \boxed{\frac{2}{7}}$

(3) $\frac{(a-b)^3(a+b)^3}{(a-b)(a+b)^2} = \boxed{(a-b)^2(a+b)}$

(4) letter

$\boxed{\frac{6!}{2!\cdot 2!}} = \frac{120\cdot 6}{2\cdot 2} = 60\cdot 3 = \boxed{180}$

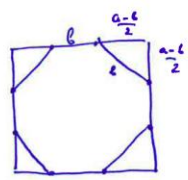
(5)



$\boxed{C = (1, \sqrt{3}) \text{ or } (1, -\sqrt{3})}$

(6) $a^4-b^4 = (a^2-b^2)(a^2+b^2) = \boxed{(a-b)(a+b)(a^2+b^2)}$

(7)



$b^2 = 2\left(\frac{a-b}{2}\right)^2$

$2b^2 = (a-b)^2$

$b^2+2ab-a^2=0 \qquad b=-a\pm\sqrt{a^2+a^2}$

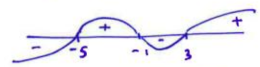
$b=(\sqrt{2}-1)a$

Area $= a^2 - 4\cdot\left(\frac{a-b}{2}\right)^2\cdot\frac{1}{2} = a^2 - \frac{(a-b)^2}{2}$

$= a^2 - \frac{a^2}{2}(2-\sqrt{2})^2 = a^2 - a^2\left[2-2\sqrt{2}+1\right] = a^2\left[2\sqrt{2}-2\right]$

$\boxed{\text{Area} = 2(\sqrt{2}-1)a^2}$

(8) $\frac{x+5}{x^2-2x-3} > 0 \qquad \frac{x+5}{(x-3)(x+1)} > 0 \qquad x\neq 3,\ x\neq -1$



$\boxed{x\in(-5,-1)\cup(3,+\infty)}$

(9) $(x+2)^{10} = x^{10}+\dots + a_4x^6+\dots \qquad a_4 = C_{10}^{4} = \frac{10!}{6!\cdot 4!}$

$a_4 = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2} = 210 \qquad \boxed{a_4 = 210}$

alternatively by Pascal's triangle

(10) $\exists\, n = 7k+2 = 13m+5$

$7k = 3 \mod 13 \qquad 7k' = 1 \mod 13$

$k'=2$

$k = 6+13l \qquad (m=7l+3) \qquad 7\cdot 6 = 3 \mod 13$

$n = 7(6+13l)+2 = 7\cdot 13\, l + 44$

$\boxed{n = 91l+44 \qquad l = 0, \pm 1, \pm 2, \dots}$

Math 9 placement test 2014

1. Open brackets and expand the following expressions
    a. $(a+b)^2 =$
    b. $(a-b)^3 =$
2. Factor the following expressions
    a. $a^2 - b^2 =$
    b. $a^3 - b^3 =$
    c. $a^3 + b^3 =$
    d. $1 + a + a^2 + a^3 =$
3. Find the coefficient of x^5 in the expression $(1+2x)^8$
4. Find the remainder of 3^{2014} upon division by 7.
5. Solve the equation
$$\frac{x^2+1}{2x} + \frac{2x}{x^2+1} = 2$$
6. Eight teams have reached the quarter-finals of the soccer World Cup.
    a. How many ways are there for these teams to be paired to play the quarter-final games?
    b. How many different outcomes of which team wins which medal (gold, silver, bronze) are possible?
7. Corners of a square with the side a are cut off so that a regular octagon is obtained. Find the area of this octagon.

Math 9 placement test 2014: solutions to selected problems

1. Open brackets and expand the following expressions
   a. $(a+b)^2 =$
   b. $(a-b)^3 =$
2. Factor the following expressions
   a. $a^2 - b^2 =$
   b. $a^3 - b^3 =$
   c. $a^3 + b^3 =$
   d. $1 + a + a^2 + a^3 =$
3. Find the coefficient of x^5 in the expression $(1+2x)^8$

$$(1+2x)^8 = \cdots + C_8^3(2x)^5 + \cdots = \cdots + \frac{8!}{5! \cdot 3!} \cdot 2^5 \cdot x^5 + \cdots = \cdots + 7 \cdot 8 \cdot 32 \cdot x^5 + \cdots$$

4. Find the remainder of 3^{2014} upon division by 7.

$3^{2014} = (7+2)^{1007} = (\dots) \cdot 7 + 2^{1007} = (\dots) \cdot 7 + 4 \cdot (7+1)^{335} = (\dots) \cdot 7 + 4$

5. Solve the equation

$$\frac{x^2+1}{2x} + \frac{2x}{x^2+1} = 2$$

$\frac{x^2+1}{2x} = t$, then $\frac{x^2+1}{2x} + \frac{2x}{x^2+1} = 2 \Leftrightarrow t + \frac{1}{t} = 2 \Leftrightarrow t^2 - 2t + 1 = 0 \Leftrightarrow t = 1 \Leftrightarrow \frac{x^2+1}{2x} = 1 \Leftrightarrow x^2 - 2x + 1 = 0 \Leftrightarrow x = 1$.

8. Eight teams have reached the quarter-finals of the soccer World Cup.
   a. How many ways are there for these teams to be paired to play the quarter-final games?

   For 2n teams, let the number of pairings be P_n. There is 2n-1 ways to pair the first team and thus select the first pair. For the remaining 2n-2 teams, there will be 2n-3 ways to select the second pair, and so on. Hence, $P_n = (2n-1) \cdot P_{n-1} = (2n-1) \cdot (2n-3) \cdot \ldots \cdot 3 \cdot 1$. For the case of 8 teams, $P_4 = 7 \cdot 5 \cdot 3 = 105$.

   b. How many different outcomes of which team wins which medal (gold, silver, bronze) are possible?

   This is given by the number of possible ways to select 3 out of 8, where order matters. The answer is $A_8^3 = \frac{8!}{5!} = 6 \cdot 7 \cdot 8 = 336$.

9. Corners of a square with the side a are cut off so that a regular octagon is obtained. Find the area of this octagon.