October 17, 2021

Geometry.

"Direct" and "Inverse" Theorems.

Each theorem consists of premise and conclusion. Premise is a proposition supporting or helping to support a conclusion.

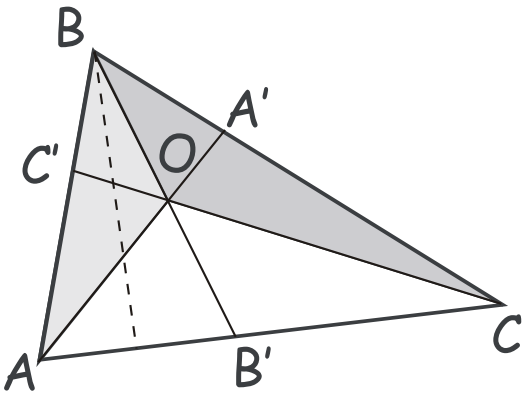
If we have two propositions, A (premise) and B (conclusion), then we can make a proposition $A \Rightarrow B$ (If A is truth, then B is also truth, A is sufficient for B, or B follows from A, or B is necessary for A). This statement is sometimes called the "direct" theorem and must be proven.

Or we can construct a proposition $A \Leftarrow B$ (A is truth only if B is also truth, A is necessary for B, or A follows from B, B is sufficient for A), which is sometimes called the "inverse" theorem, and also must be proven.

While some theorems offer only necessary or only sufficient condition, most theorems establish equivalence of two propositions, $A \Leftrightarrow B$.

Ceva's Theorem.

Definition. Cevian is a line segment in a triangle, which joins a vertex with a point on the opposite side.



Theorem (Ceva). In a triangle ABC, three cevians AA', BB', and CC' are concurrent (intersect at a single point O) **if and only if**

$$\frac{|AB'|}{|B'C|} \cdot \frac{|CA'|}{|A'B|} \cdot \frac{|BC'|}{|C'A|} = 1$$

This theorem was published by Giovanni Ceva in his 1678 work De lineis rectis.

Direct Ceva's theorem. Geometrical proof.

For the Ceva's theorem the premise (A) is "Three Cevians in a triangle ABC, AA', CC', BB', are concurrent". The conclusion (B) is,

$\frac{|AC'|}{|C'B|}\cdot\frac{|BA'|}{|A'C|}\cdot\frac{|CB'|}{|B'A|}=1$. The full statement of the "direct" theorem is $A\Rightarrow B$, i.e.,

If three cevians in a triangle ABC, AA', CC', BB', are concurrent, **then** $\frac{|AC'|}{|C'B|}\cdot\frac{|BA'|}{|A'C|}\cdot\frac{|CB'|}{|B'A|}=1$ is true. From A follows B, $A\ \Rightarrow B$. Again, premise in the "direct" theorem provides sufficient condition for the conclusion to hold. Clearly, the conclusion B is the necessary condition for the premise A to hold.

Proof. Consider triangles AOB, BOC and COA. Denote their areas S_{AOB}, S_{BOC}, and S_{COA}. The trick is to express the desired ratios of the lengths of the 6 segments, $|AB'|:|B'C|$, $|CA'|:|A'B|$, $|BC'|:|C'A|$, in terms of the ratios of these areas. We note that some triangles share altitudes. Therefore,

$$\frac{|AB'|}{|B'C|}=\frac{S_{ABB'}}{S_{B'BC}};\ \frac{|AB'|}{|B'C|}=\frac{S_{AOB'}}{S_{B'OC}},\text{ and so on.}$$

The above two equalities yield,

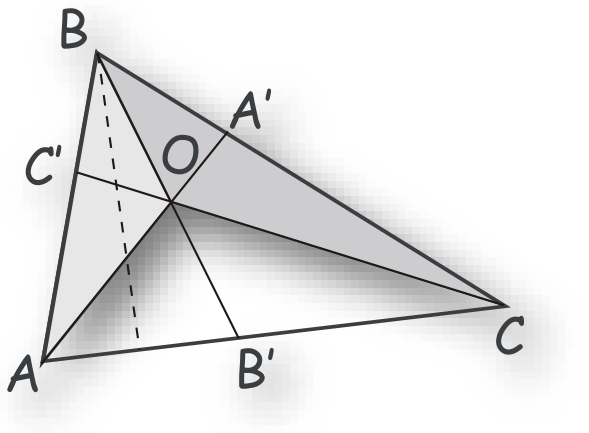
$$\frac{|AB'|}{|B'C|}=\frac{S_{ABB'}-S_{AOB'}}{S_{B'BC}-S_{B'OC}}=\frac{S_{AOB}}{S_{BOC}}$$

Repeating this for the other ratios along the sides of the triangle we obtain,

$$\frac{|AB'|}{|B'C|}\cdot\frac{|CA'|}{|A'B|}\cdot\frac{|BC''|}{|C'A|}=\frac{S_{AOB}}{S_{BOC}}\cdot\frac{S_{AOC}}{S_{BOA}}\cdot\frac{S_{BOC}}{S_{COA}}=1,$$

which completes the proof.

"Inverse" Ceva's theorem. Geometrical proof.



Let us formulate the "inverse Ceva's theorem", the theorem where premise and conclusion switch places.

If in a triangle ABC three chevians divide sides in such a way that

$$\frac{|AC'|}{|C'B|}\cdot\frac{|BA'|}{|A'C|}\cdot\frac{|CB'|}{|B'A|}=1 \qquad (1)$$

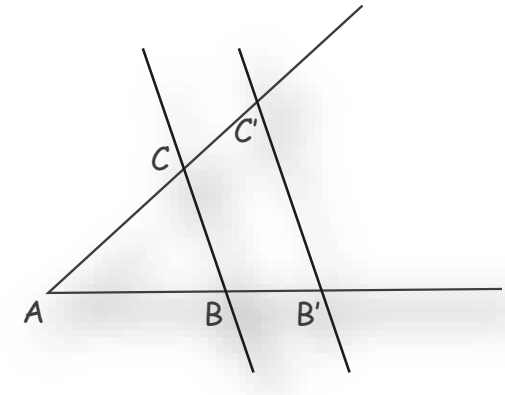
holds, **then** they are concurrent. A follows from B, $B\Rightarrow A$, or $A\Leftarrow B$, or, $\sim A\Rightarrow\sim B$, in other words if the three cevians of a triangle ABC are not concurrent, then $\frac{|AC'|}{|C'B|}\cdot\frac{|BA'|}{|A'C|}\cdot\frac{|CB'|}{|B'A|}\neq 1$. Three cevians being concurrent is a necessary condition for the relation $\frac{|AC'|}{|C'B|}\cdot\frac{|BA'|}{|A'C|}\cdot\frac{|CB'|}{|B'A|}=1$ to hold.

Proof. An inverse theorem can often be proven by contradiction (reductio ad absurdum), assuming that it does not hold and arriving at a contradiction with the already proven direct theorem. Assume that Eq. (1) holds, but one of the cevians, say BB', does not pass through the intersection point, O, of the other two cevians. Let us then draw another cevian, BB'', which passes through O. By direct Ceva theorem we have then, $\frac{|CB''|}{|B''A|}=\frac{|C'B|}{|AC'|}\cdot\frac{|A'C|}{|BA'|}=\frac{|CB'|}{|B'A|}$, which means that B' and B''coincide, and therefore AB', must pass through O.

Thus, in the case of Ceva's theorem premise and conclusion (propositions A and B) are equivalent, ($A\Leftrightarrow B$), and we can state the theorem as follows

Theorem (Ceva). Three cevians in a triangle ABC, AA', CC', BB', are concurrent**, if and only if** $\frac{|AC'|}{|C'B|}\cdot\frac{|BA'|}{|A'C|}\cdot\frac{|CB'|}{|B'A|}=1$.

"Inverse" Thales theorem.



The "inverse" Thales theorem states

If lengths of segments in the Figure on the left satisfy $\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}$, **then** lines BC and BC' are parallel. The proof is similar to the proof of Ceva's "inverse" theorem, by assuming the opposite and obtaining a contradiction.

If a theorem establishes the equivalence of two propositions A and B, $A \Leftrightarrow B$, it is actually often the case that the proof of the necessary condition, $A \Leftarrow B$, i. e. the "inverse" theorem, is much simpler than the proof of the "direct" proposition, establishing the sufficiency, $A \Rightarrow B$. It often could be achieved by using the sufficiency condition which has already been proven, and employing the method of "proof by contradiction", or another similar construct.

<u>Examples of necessary and sufficient statements</u>

- Predicate A: "quadrilateral is a square"

  Predicate B: "all four its sides are equal"

  Which of the following holds: $A \Rightarrow B$, $A \Leftarrow B$, $A \Leftrightarrow B$?

  Is A necessary or sufficient condition for B?

  If a quadrilateral is not square its four sides are not equal. Truth or not? ($A \Leftarrow B$ or $\sim A \Rightarrow \sim B$).

- Predicate A:

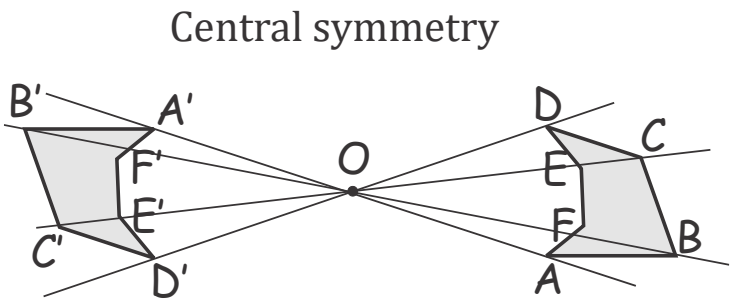
  Predicate B:

  Which of the following holds: $A \Rightarrow B$, $A \Leftarrow B$, $A \Leftrightarrow B$?

Similarity and homothety.

Recap: Central Symmetry.

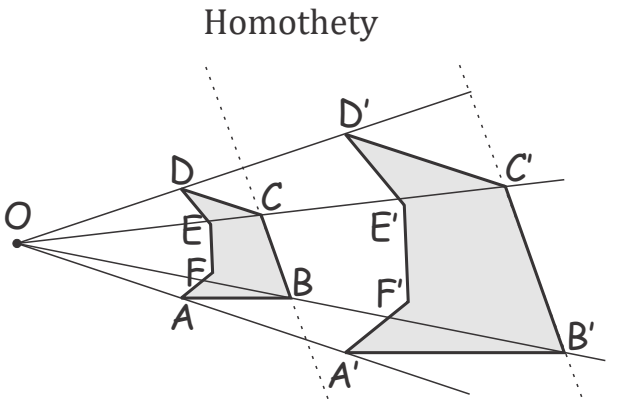
Definition. Two points A and A' are symmetric with respect to a point O, if O is the midpoint of the segment AA'.



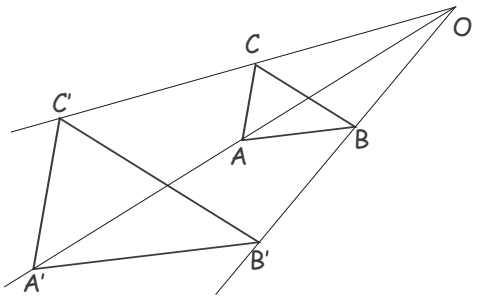
Definition. Two figures are symmetric with respect to a point O, if for each point of one figure there is a symmetric point belonging to the other figure, and vice versa. The point O is called the center of symmetry.

Symmetric figures are congruent and can be made to coincide by a 180 degree rotation of one of the figures around the center of symmetry.

Recap: Similarity and homothety.



Definition. Two figures are homothetic with respect to a point O, if for each point A of one figure there is a corresponding point A' belonging to the other figure, such that A' lies on the line (OA) at a distance $|OA'| = k|OA|$ $(k > 0)$ from point O, and vice versa, for each point A' of the second figure there is a corresponding point A belonging to the first figure, such that A' lies on the line (OA) at a distance $|OA| = \frac{1}{k}|OA'|$ from point O. Here the positive number k is called the homothety (or similarity) coefficient. Homothetic figures are **similar**. The transformation of one figure (e.g. multilateral $ABCDEF$) into the figure $A'B'C'D'E'F'$ is called homothety, or similarity transformation.



Thales Theorem Corollary 1. The corresponding segments (e.g. sides) of the homothetic figures are parallel.

Thales Theorem Corollary 2. The ratio of the corresponding elements (e.g. sides) of the homothetic figures equals k.

Exercise. What is the ratio of the areas of two similar (homothetic) figures?

Definition. Consider triangles, or polygons, such that angles of one of them are congruent to the respective angles of the other(s). Sides which are adjacent to the congruent angles are called *homologous*. In triangles, sides opposite to the congruent angles are also homologous.

Definition. Two triangles are similar if (i) angles of one of them are congruent to the respective angles of the other, or (ii) the sides of one of them are proportional to the homologous sides of the other.

Arranging 2 similar triangles, so that the intercept theorem can be applied

The similarity is closely related to the intercept (Thales) theorem. In fact this theorem is equivalent to the concept of similar triangles, i.e. it can be used to prove the properties of similar triangles, and similar triangles can be used to prove the intercept theorem. By matching identical angles one can always place 2 similar triangles in one another, obtaining the configuration in which the intercept theorem applies and vice versa the intercept theorem configuration always contains 2 similar triangles. In particular, a line parallel to any side of a given triangle cuts off a triangle similar to the given one.

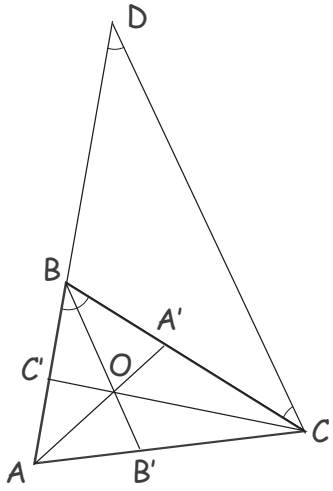
Similarity tests for triangles.

- Two angles of one triangle are respectively congruent to the two angles of the other

- Two sides of one triangle are proportional to the respective two sides of the other, and the angles between these sides are congruent
- Three sides of one triangle are proportional to three sides of the other

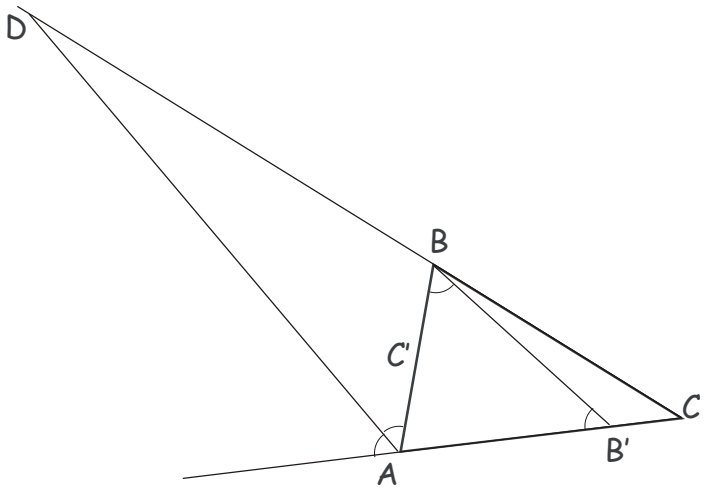
Theorem (property of the bisector). The bisector of any angle of a triangle divides the opposite side into parts proportional to the adjacent sides,

$$\frac{|AC\prime|}{|C\prime B|} = \frac{|AC|}{|BC|}, \frac{|BA\prime|}{|A\prime C|} = \frac{|AB|}{|AC|}, \frac{|CB\prime|}{|B\prime A|} = \frac{|BC|}{|AB|}$$



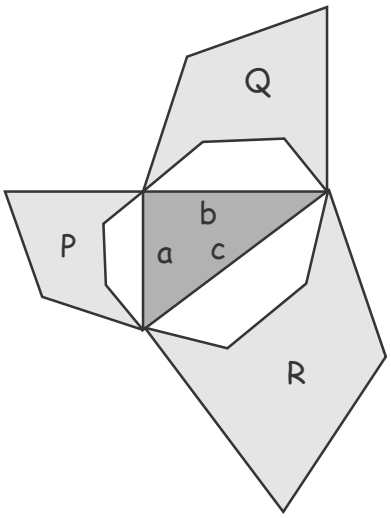
Proof. Consider the bisector BB'. Draw line parallel to BB' from the vertex C, which intercepts the extension of the side AB at a point D. Angles $B'BC$ and BCD have parallel sides and therefore are congruent. Similarly, are congruent ABB' and CDB. Hence, triangle CBD is isosceles, and $|BD| = |BC|$. Now, applying the intercept theorem to the triangles ABB' and ACD, we obtain $\frac{|CB\prime|}{|B\prime A|} = \frac{|BD|}{|AB|} = \frac{|BC|}{|AB|}$.

Theorem (property of the external bisector). The bisector of the exterior angle of a triangle intercepts the opposite side at a point (D in the Figure) such that the distances from this point to the vertices of the triangle belonging to the same line are proportional to the lateral sides of the triangle.



Proof. Draw line parallel to AD from the vertex B, which intercepts the side AC at a point B'. Angles ABB' and DAB have parallel sides and therefore are congruent. Similarly, we see that angles $AC'B$ and ABB' are congruent, and, therefore, $|AB'| = |AB|$. Applying the intercept theorem, we obtain, $\frac{|DB|}{|DC|} = \frac{|AB\prime|}{|AC|} = \frac{|AB|}{|AC|}$.

Generalized Pythagorean Theorem.



Theorem. If three similar polygons, P, Q and R with areas S_P, S_Q and S_R are constructed on legs a, b and hypotenuse c, respectively, of a right triangle, then,

$$S_P + S_Q = S_R$$

Proof. The areas of similar polygons on the sides of a right triangle satisfy $\frac{S_R}{S_P} = \frac{c^2}{a^2}$ and $\frac{S_R}{S_Q} = \frac{c^2}{b^2}$, or, $\frac{S_P}{a^2} = \frac{S_Q}{b^2} = \frac{S_R}{c^2}$. Using the property of a proportion, we may then write, $\frac{S_P+S_Q}{a^2+b^2} = \frac{S_R}{c^2}$, wherefrom, using the Pythagorean theorem for the right triangle, $a^2 + b^2 = c^2$, we immediately obtain $S_P + S_Q = S_R$.

Exercise. Show that for any proportion,

$$\left(\frac{a}{b} = \frac{c}{d}\right) \Rightarrow \left(\frac{a+c}{b+d} = \frac{a}{b} = \frac{c}{d}\right) \wedge \left(\frac{a-c}{b-d} = \frac{a}{b} = \frac{c}{d}, if\ b \neq d\right)$$