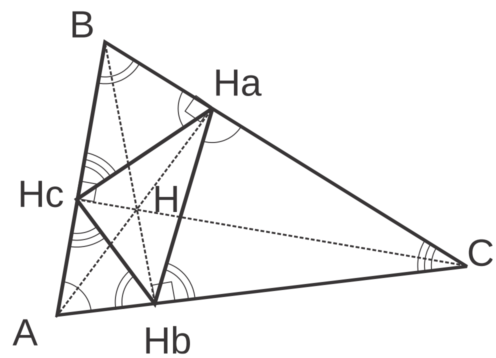
October 31, 2021

Geometry.

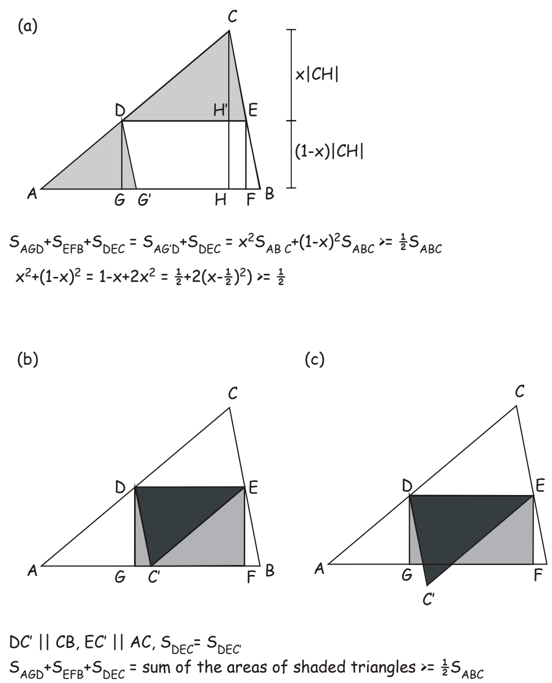
<u>**Selected problems on similar triangles (from last homeworks)**</u>.

Problem 1(5). Prove that altitudes of any triangle are the bisectors in another triangle, whose vertices are the feet of these altitudes (hint: prove that the line connecting the feet of two altitudes of a triangle cuts off a triangle similar to it).



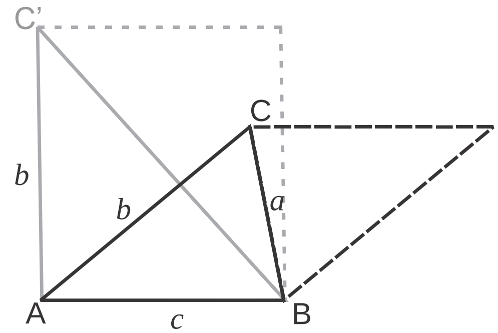
Solution. Notice similar right triangles, $ACH_a \sim BCH_b$, which implies, $\frac{|AC|}{|BC|} = \frac{|CH_a|}{|CH_b|}$. Therefore, $CH_aH_b \sim ABC$. Similarly, from $CAH_c \sim BAH_b$ it follows that $AH_bH_c \sim ABC$, and from $ABH_a \sim BCH_c$ that $BH_cH_a \sim ABC$.

Problem 2(2). Rectangle DEFG is inscribed in triangle ABC such that the side DE belongs to the base AB of the triangle, while points F and G belong to sides BC abd CA, respectively. What is the largest area of rectangle DEFG?



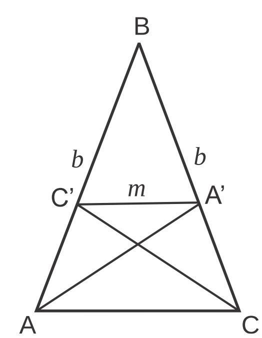
Solution. Notice similar triangles, $CDE \sim ABC$, wherefrom the vertical side of the rectangle is, $|DG| = |EF| = |CH| - |CH'| = \left(1 - \frac{|DE|}{|AB|}\right)|CH|$, so that the area of the rectangle is, $S_{DEFG} = |DE||DG| = |DE|\left(1 - \frac{|DE|}{|AB|}\right)|CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)|AB||CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)2S_{ABC}$. Using the geometric-arithmetic mean inequality, $\frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right) \leq \left(\frac{\frac{|DE|}{|AB|}+1-\frac{|DE|}{|AB|}}{2}\right)^2 = \frac{1}{4}$, where the largest value of the left side is achieved when $\frac{|DE|}{|AB|} = 1 - \frac{|DE|}{|AB|}$, and therefore $S_{DEFG} = \frac{1}{2}S_{ABC}$. There are a number of other possible solutions, some of which are shown in the figures.

Problem 3(1). Prove that for any triangle ABC with sides a, b and c, the area, $S \le \frac{1}{4}(b^2+c^2)$.



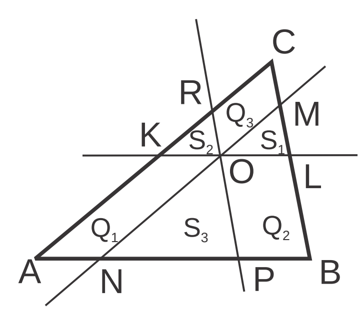
Solution. Notice that of all triangles with given two sides, b and c, the largest area has triangle ABC', where the sides with the given lengths, $|AB| = c$ and $|AC| = b$ form a right angle, $\widehat{BAC} = 90°$ (b is the largest possible altitude to side c). Therefore, $\forall \Delta ABC, S_{ABC} \le S_{ABC'} = \frac{1}{2}bc \le \frac{1}{2}\frac{b^2+c^2}{2}$, where the last inequality follows from the arithmetic-geometric mean inequality, $bc \le \frac{b^2+c^2}{2}$ (or, alternatively, follows from $b^2+c^2-2bc = (b-c)^2 \ge 0$.

Problem 4(2). In an isosceles triangle ABC with the side $|AB| = |BC| = b$, the segment $|A'C'| = m$ connects the intersection points of the bisectors, AA' and CC' of the angles at the base, AC, with the corresponding opposite sides, $A' \in BC$ and $C' \in AB$. Find the length of the base, $|AC|$ (express through given lengths, b and m).



Solution. From Thales proportionality theorem we have, $\frac{|AC|}{m} = \frac{|BC|}{|BA'|} = \frac{|BA'|+|A'C|}{|BA'|} = 1 + \frac{|A'C|}{|BA'|} = 1 + \frac{|AC|}{b}$, where we have used the property of the bisector, $\frac{|A'C|}{|BA'|} = \frac{|AC|}{|AB|} = \frac{|AC|}{b}$. We thus obtain, $|AC| = \frac{1}{\frac{1}{m}-\frac{1}{b}} = \frac{bm}{b-m}$.

Problem 5(5). Three lines parallel to the respective sides of the triangle ABC intersect at a single point, which lies inside this triangle. These lines split the triangle ABC into 6 parts, three of which are triangles with areas S_1, S_2, and S_3. Show that the area of the triangle ABC, $S = \left(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3}\right)^2$ (see Figure).



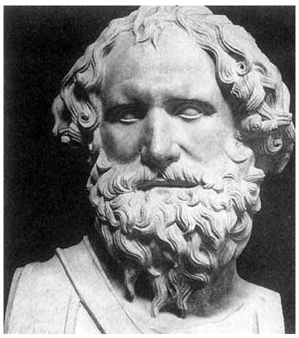
Solution. Denote $\frac{S_1}{S} = k_1, \frac{S_2}{S} = k_2, \frac{S_3}{S} = k_3$. Then, $\frac{S_1+S_2+Q_3}{S} = k_1+k_2+\frac{Q_3}{S} = \left(\sqrt{k_1}+\sqrt{k_2}\right)^2$, so, $Q_3 = 2S\sqrt{k_1k_2} = \sqrt{S_1S_2}$, $Q_2 = \sqrt{S_3S_1}$, $Q_1 = \sqrt{S_2S_3}$.

The Law of Lever. The Method of the Center of Mass.

Archimedes' Law of Lever.

"Give me a place to stand on, and I will move the earth."

quoted by Pappus of Alexandria in Synagoge, Book VIII, c. AD 340



Archimedes of Syracuse

Born c. 287 BC

Syracuse, Sicily

Magna Graecia

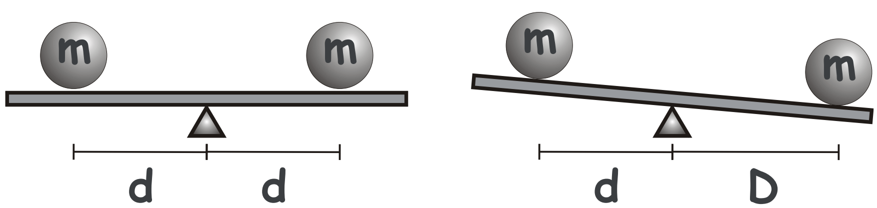
Died c. 212 BC (aged around 75) , Syracuse

Archimedes of Syracuse generally considered the greatest mathematician of antiquity and one of the greatest of all time. Archimedes anticipated modern calculus and analysis by applying concepts of infinitesimals and the method of exhaustion to derive and rigorously prove a range of geometrical theorems, including the area of a circle, the surface area and volume of a sphere, and the area under a parabola.

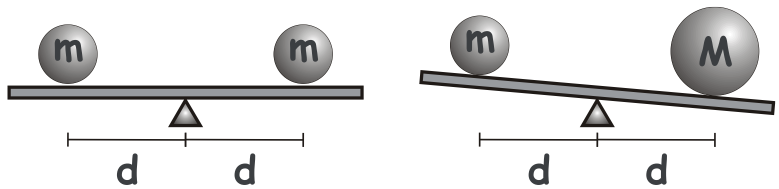
He was also one of the first to apply mathematics to physical phenomena, founding hydrostatics and statics, including an explanation of the principle of the lever. He is credited with designing innovative machines, such as his screw pump, compound pulleys, and defensive war machines to protect his native Syracuse from the Roman invasion.

Archimedes derives the Law of Lever from several simple axioms (assumptions), which summarize the everyday experience, in a manner similar to those in Euclidean geometry.

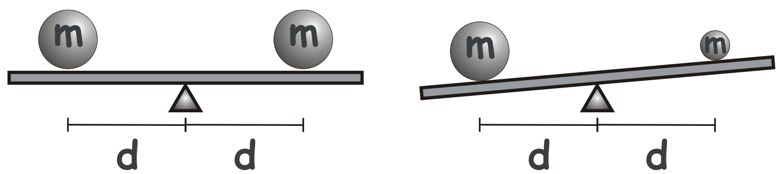
Axiom 1. Equal weights at equal distances from the fulcrum balance. Equal weights at unequal distance from the fulcrum do not balance, but the weight at the greater distance will tilt its end of the lever down.



Axiom 2. If, when two weights balance, we add something to one of the weights, they no longer balance. The side holding the weight we increased goes down.



Axiom 3. If, when two weights balance, we take something away from one of them, they no longer balance. The side holding the weight we did not change goes down.



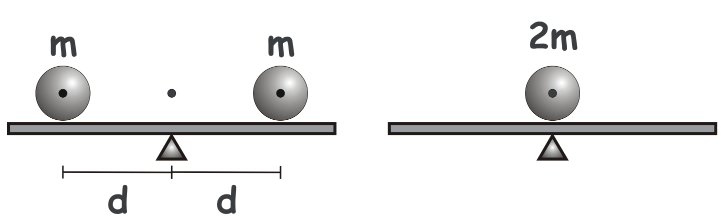
Archimedes then proves the inverse statements as propositions (theorems).

Proposition 1. Weights that balance at equal distances from the fulcrum are equal.

Proposition 2. Unequal weights at equal distances from the fulcrum do not balance, but the side holding the heavier weight goes down.

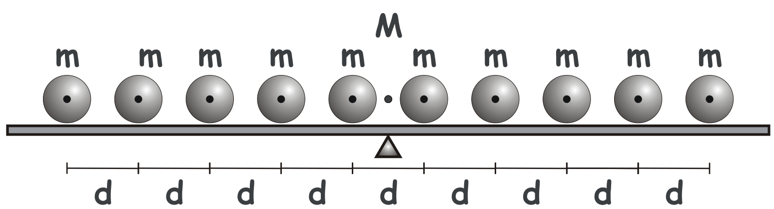
Proposition 3. Unequal weights balance at unequal distances from the fulcrum, the heavier weight being at the shorter distance.

Proposition 4. If two equal weights have different centers of gravity then the center of gravity of the two together is the midpoint of the line segment joining their centers of gravity.



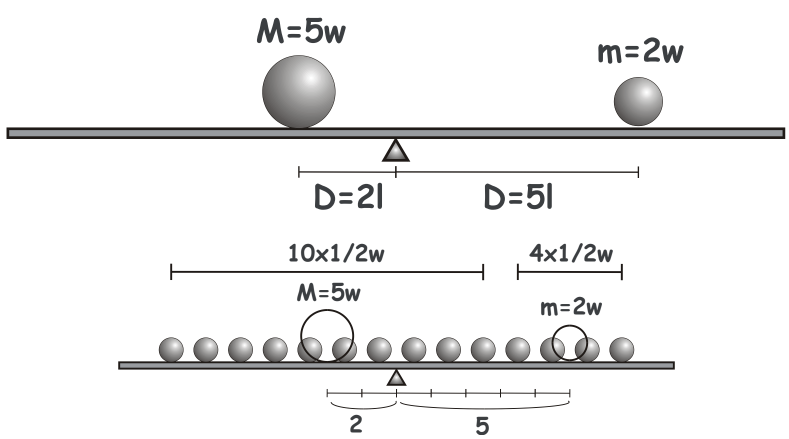
Proposition 4 is just a rephrase of the Axiom 1, where Archimedes tacitly introduces the notion of the **Center of Gravity** (**Center of Mass**). The way to understand the Proposition 4 is to treat the entire weight as if it is located at a single point, its center of gravity. In other words, we can picture each weight (mass) as concentrated in a single point, i. e. as a **Point Mass**. We shall use terms weight and mass interchangeably, assuming that weight is associated with a mass in the homogeneous gravitation field, and therefore is proportional to the mass. The following observation immediately follows from the Proposition 4.

Corollary. If an even number of equal weights have their centers of gravity situated along a straight line such that the distances between the consecutive weights are all equal, then the center of gravity of the entire system is the midpoint of the line segments joining the centers of gravity of the two weights in the middle.



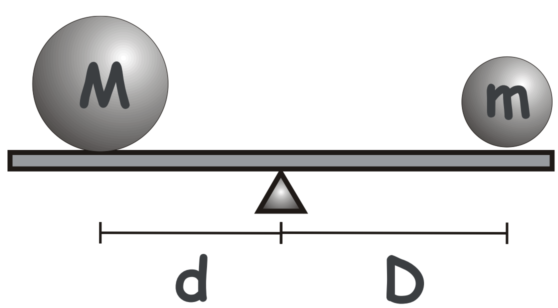
At this point Archimedes proves the Law of Lever, first only for commensurate weights.

Proposition 5. Commensurate weights (masses) balance at distances from the fulcrum, which are inversely proportional to their magnitudes, $\frac{d}{D} = \frac{M}{m}$.



Proof. Let w be the greatest common measure of weights (masses) m and M, $m = pw$, $M = nw$, $\{p, n\} \in N$. Let us split weight M into $2n$ smaller pieces, each of weight $w/2$, and weight m into $2p$ smaller pieces of weight $w/2$. Let us now split the segment connecting M and m into $n + p$ congruent smaller segments, and also mark n such segments on the opposite side of weight M and p such segments on the opposite side of weight m. Let us now place all $2(n + p)$ smaller weights w at the centers of these $2(n + p)$ segments as shown in the Figure. Clearly, since each of the initial weights was split into an even number of equal pieces, which were placed symmetrically around its initial position, the resultant system of smaller weights has the same center of gravity as the original weight. On the other hand, the obtained system of $2(n + p)$ weights $w/2$ has the center of gravity in the middle, at a distance of p segments from the position of weight M and n segments from the position of weight m, as illustrated in the Figure. Therefore, $\frac{D}{d} = \frac{p}{n} = \frac{m}{M}$, which proves the Law of Lever for the commensurate weights. The theorem for the incommensurate weights is then proven by reducing to contradiction.

Theorem (Law of Lever). Masses (weights) balance at distances from the fulcrum, which are inversely proportional to their magnitudes,



$$\frac{D}{d} = \frac{M}{m} \Leftrightarrow Md = mD$$

For commensurate masses, $= p \cdot w, m = q \cdot w,\ \ p, q \in \mathbb{N}$, the Law was proven using the main “trick” of the mass points method: each of the two masses is split into $2p$ and $2q$ smaller masses, $w/2$, respectively, which are then re-positioned in pairs around the original masses so that positions of the center of mass (COM) for each of the two original masses do not change, but the COM position for the whole system becomes obvious.

In order to prove the Law of Lever for incommensurate masses, we first make the following observation.

Lemma. If two commensurate masses m and M are placed at distances D and d from the fulcrum, respectively, then M goes up if and only if $Md < mD$,

$$(M\ rises\ up) \Leftrightarrow (Md < mD)$$

First, if distances d and D are incommensurate, we move mass M slightly, to a position d' which is commensurate with D, but such that M still rises up. Therefore, we only need to consider case when d and D are commensurate. Since M rises up, we need to increase mass M to achieve balance. Let $M' > M$ be such that M' and m balance. Using the Law of Lever for commensurate masses we have, $M' = m\frac{D}{d}$ (because distances are commensurate, so are the masses). Since $M < M' = m\frac{D}{d}$, it follows that $Md < mD$. Conversely, if $Md < mD$ we can increase it to $M' = m\frac{D}{d}$, which balances m. Decreasing mass from back to M will cause it to rise.

Corollary. The converse statement immediately follows via excluded middle,

$$(M\ goes\ down) \Leftrightarrow (Md > mD)$$

Proof (case of incommensurate masses). Let now two incommensurate masses m and M, be placed at distances d and D from the fulcrum, respectively, such that the Law of Lever is satisfied, $Md = mD$. Assume that the masses nevertheless do not balance, for example, M goes down. Decrease mass M by a small amount, turning it into M', such that it still goes down, but is now commensurate with m. Now m and M' are commensurate, and $mD > M'd$, which means that M' should rise. This contradicts our assumption, so m and M must balance. Note that in the above we used a non-trivial fact that a commensurate mass, or distance can be found that differs from the given incommensurate one by an arbitrarily small amount. This means that for any irrational number there exists a rational number, which differs from it as little as we want, i. e. that rational numbers are dense.

Method of the Center of Mass (Mass Points).

Definition. For two point masses, m_A and m_B at points A and B, the center of mass lies at a point C' on the straight line segment $|AB|$, such that,

$$\frac{|AC'|}{|C'B|} = \frac{m_B}{m_A}.$$

When finding the center of mass in a system of point masses, one can replace any pair of masses, m_A and m_B, with a single point mass having the total mass $m_A + m_B$, placed at the center of mass of the pair.

The following important properties of the Center of Mass follow immediately.

1. Every system of finite number of point masses has unique center of mass (COM).
2. For two point masses, the COM belongs to the segment connecting these points; its position is determined by the Archimedes lever rule: the point's mass times the distance from it to the COM is the same for both points.
3. The position of the system's center of mass does not change if we move any subset of point masses in the system to the center of mass of this subset. In other words, we can replace any number of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.

Ceva's Theorem: Point Masses.

We select masses, m_A, m_B, and m_C such that the corresponding centers of mass for each pair are at points A', B' and C', respectively. Then,

$$\frac{|AB'|}{|B'C|} \cdot \frac{|CA'|}{|A'B|} \cdot \frac{|BC''|}{|C'A|} = \frac{m_C}{m_A} \cdot \frac{m_B}{m_C} \cdot \frac{m_A}{m_B} = 1.$$