November 14, 2021

Algebra.

<u>Solutions to some homework problems</u>.

1. Using mathematical induction, prove that $\forall n \in \mathbb{N}$,
   a. $\sum_{k=1}^{n}(2k-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{4n^3-n}{3}$,
   b. $\sum_{k=1}^{n}(2k)^2 = 2^2+4^2+6^2+\cdots+(2n)^2 = \frac{2n(2n+1)(n+1)}{3}$
   c. $\sum_{k=1}^{n}k^3 = 1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2$
   d. $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)} = \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)} < \frac{1}{2}$
   e. $\sum_{k=1}^{n}\frac{1}{(7k-6)(7k+1)} = \frac{1}{1\cdot 8}+\frac{1}{8\cdot 15}+\frac{1}{15\cdot 22}+\cdots+\frac{1}{(7n-6)(7n+1)} < \frac{1}{7}$
   f. $\sum_{k=n+1}^{3n+1}\frac{1}{k} = \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{3n+1} > 1$

Solution of (f)

<u>Basis</u>: $P_1: \sum_{k=2}^{4}\frac{1}{k} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1$

<u>Induction</u>: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=n+2}^{3n+4}\frac{1}{k} = \frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{3n+4} > 1$

<u>Proof</u>: $\sum_{k=n+2}^{3n+4}\frac{1}{k} = \frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{3n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}+\frac{1}{3n+4} = \sum_{k=n+1}^{3n+1}\frac{1}{k} + \frac{1}{3n+2}+\frac{1}{3n+3}+\frac{1}{3n+4}-\frac{1}{n+1} > 1$, because $\sum_{k=n+1}^{3n+1}\frac{1}{k} > 1$ by induction assumption, and $\frac{1}{3n+2}+\frac{1}{3n+3}+\frac{1}{3n+4}-\frac{1}{n+1} = \frac{1}{3}\left(\frac{1}{n+\frac{2}{3}}+\frac{1}{n+\frac{4}{3}}-\frac{2}{n+1}\right) = \frac{1}{3}\left(\frac{2n+2}{\left(n+\frac{2}{3}\right)\left(n+\frac{4}{3}\right)}-\frac{2}{n+1}\right) \geq \frac{1}{3}\left(\frac{2n+2}{(n+1)^2}-\frac{2}{n+1}\right) \geq 0$ (here we used the arithmetic-geometric mean inequality, $\sqrt{\left(n+\frac{2}{3}\right)\left(n+\frac{4}{3}\right)} \leq \frac{2n+2}{2} = n+1$).

2. Problems on binomial coefficients, which are defined as,
   $C_n^k = {}_kC_n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   a. Prove that $C_{n+k}^2 + C_{n+k+1}^2$ is a full square
   b. Find n satisfying the following equation,

$$C_n^{n-1} + C_n^{n-2} + C_n^{n-3} + \cdots + C_n^{n-10} = 1023$$

c. Prove that

$$\frac{C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n}{n} = 2^{n-1}$$

Solution of (b)

$C_n^{n-1} + C_n^{n-2} + C_n^{n-3} + \cdots + C_n^{n-10} = C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^{10} = C_n^0 + C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^{10} - 1$, so, $C_n^0 + C_n^1 + C_n^2 + C_n^3 + \cdots + C_n^{10} = 1024 = 2^{10}$, which is satisfied for $n = 10$ thanks to the property of the binomial coefficients,

$$C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^k + \cdots + C_n^n = (1+1)^n = 2^n$$

Solution of (c)

$$\frac{C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n}{n} = C_{n-1}^0 + C_{n-1}^1 + C_{n-1}^2 + \cdots + C_{n-1}^{n-1} = 2^{n-1}$$

Elements of Set Theory.

Definition. We will define a **set** to be a group of objects (not necessarily ordered) with no duplicates.

Note that the objects in the sets can themselves be sets. We can describe a set by defining some property of objects in it. For example,

1. the set containing the positive integers from 1 to 5 is $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$
2. the set of all natural integers, which we denote $\mathbb{N}$
3. the set of all integer numbers, which we denote $\mathbb{Z}$
4. the set of all rational numbers, $\frac{\boldsymbol{m}}{\boldsymbol{n}}$, $(\{\boldsymbol{m}, \boldsymbol{n}\} \in \mathbb{Z} \wedge \boldsymbol{n} \neq \mathbf{0})$, which we denote $\mathbb{Q}$
5. the set of all real numbers, which we denote $\mathbb{R}$
6. the set of all irrational numbers, which we denote **D**

If a set has finite number of objects, it is said to be finite. Otherwise, it is infinite. The number of elements, $\boldsymbol{n}$, in a finite set $\boldsymbol{A}$, is denoted $|\boldsymbol{A}| = \boldsymbol{n}$. If elements in the set can be counted by assigning a natural integer to each element, the set is called countable. The set that is not countable is called uncountable.

Exercise. Give examples of infinite, countable, uncountable sets.

If we wish to describe an infinite set, such as the set of even positive integers, we use what is called "set builder notation".

$$M = \{x : (x \in \mathbb{Z}) \wedge (x > 0) \wedge (x/2 \in \mathbb{Z})\}$$

This is read verbally as "the set of all x such that x is integer and greater than 0 and x divided by 2 is also integer". Another example,

$$F = \{n^2 - 4 : (n \in \mathbb{Z}) \wedge (0 \leq n \leq 19)\}$$

"F is the set of all numbers of the form $n^2 - 4$, such that n is a whole number in the range from 0 to 19 inclusive", where the colon ":" is read "such that".

If x is a member of a set M, we will use notation $x \in M$, if y is not a member of a set M we will write $y \notin M$. For example statement "$0 \leq x \leq 1$" can be written as $x \in [0,1]$. Another example: If $x > 3$ and $x < 5$, so $x \in (-\infty, 5[\cap]3, +\infty) \Leftrightarrow x \in]3,5[$.

Exercise. Find the set of all values of x for which the following expression makes sense: $\sqrt{25 - x^2} = \frac{4}{x-2}$.

<u>**The algebra of sets**</u>.

An **algebraic structure** (algebra) is formed by a set of objects supplemented by a set of operations, which act on the elements of this set and obey certain algebraic laws. Typical example of an algebra are binary operations of addition and multiplication on a set of real, or integer numbers, which combine two elements to produce a third. These operations obey certain laws, such as commutative, associative, and distributive. Another example would be a set of all possible rotations of a solid body, with multiplication defined as combination of two consecutive rotations (Lie algebra, it is associative, but not commutative). The algebra of sets is an algebraic structure consisting of operations on sets (the elements of the set of sets).

Definition. An identity element (or neutral element) with respect to a binary operation on a set is an element of that set, which leaves other elements unchanged when combined with them. An identity with respect to binary addition is called an additive identity (often denoted as 0) and an identity in the case of multiplication a multiplicative identity (often denoted as 1).

Definition. The **empty set** (or **null set**) is the set which contains no objects and is denoted {}, or by the symbol $\varnothing$.

Definition. The **universal set** I (the Universe of discourse) is the set which contains all objects of any nature, and of which all other sets are subsets.

In the algebra of sets, the empty set and the universal set play roles of the additive and the multiplicative identity, respectively.

Definition. The set A is said to be a **subset** of the set B if there is no element in A that is not also in B. It is denoted by $A \subset B$, or $B \supset A$.

Exercise. Let A be a finite set, with the number of elements $|A| = n$. How many different subsets does A have (including the empty subset and A itself)?

Comparing sets.

If both statement $A \subset B$ and $B \subset A$ hold, then sets A and B are equal, $A = B$. In this case sets A and B contain exactly the same elements. The relation $A \subset B$ has some similarities with the $a \leq b$ relation between the real numbers. In particular, the following set comparison rules hold:

1. $A \subset A$
2. If $A \subset B$ and $B \subset A$ then $A = B$
3. If $A \subset B$ and $B \subset C$ then $B \subset C$
4. $\emptyset \subset A$ for any set A
5. $A \subset \mathrm{I}$ for any set A

The difference between the order relation $A \subset B$ between sets and the $\leq$ relation between real numbers is that for numbers either $a \leq b$, or $a \geq b$ always holds, while this is not necessarily the case for sets order relation.

Definition. The **union** of two sets A and B is the set of elements, which are in A **or** in B **or** in both. It is denoted by $A \cup B$ and is read 'A union B'.

Definition. The **intersection** of two sets A and B is the set of elements, which are in A **and** in B. It is denoted by $A \cap B$ and is read 'A intersection B'.

We can associate the union with the "logical sum" of sets A and B,

$A \cup B = A + B,$

and the intersection with the "logical product",

$A \cap B = A \cdot B.$

Using these definitions, it can be easily verified that these operations satisfy the following rules.

6. $A + B = B + A$
7. $A \cdot B = B \cdot A$
8. $A + (B + C) = (A + B) + C$
9. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
10. $A + A = A$
11. $A \cdot A = A$

12. $A \cdot (B + C) = (A \cdot B + A \cdot C)$
13. $A + (B \cdot C) = (A + B) \cdot (A + C)$
14. $A + \varnothing = A$
15. $A \cdot I = A$
16. $A + \mathrm{I} = I$
17. $A \cdot \varnothing = \varnothing$
18. $A \subset B$ is equivalent to either of the two, $A + B = B$, or $A \cdot B = A$

Definition. The **complement** of set A in I is the set A', which consists of all objects in I which are not in A.

The operation of obtaining a complement A' has no analogs in the algebra of numbers, and possesses the following properties.

19. $A + A' = I$
20. $A \cdot \mathrm{A}' = \varnothing$
21. $\varnothing' = I$
22. $I' = \varnothing$
23. $A'' = A$
24. $(A \subset B) \Leftrightarrow (B' \subset A')$
25. $(A + B)' = A' \cdot B'$
26. $(A \cdot B)' = A' + B'$

These 26 laws of the algebra of sets possess an interesting <u>duality</u> symmetry: if we interchange $\subset$ and $\supset$, $+$ and $\cdot$, and $\varnothing$ and I, the same set of rules is obtained. Each of the 26 relations transforms in some other of these relations.

Exercise. Verify the above stated duality.

It is also remarkable from the point of view of the axiomatic constructions that all the above 26 laws, as well as all other theorems of set algebra can be deduced from the following three equation adopted as axioms, much like the Euclidian geometry.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $(A' + B')' + (A' + B)' = A$

The operations $A \cdot B$ and $A \subset B$ are then defined by: $A \cdot B = (A' + B')'$ and $A \subset B$ means that $A + B = B$.

Exercise. Verify that all 26 rules of the set algebra can be obtained from the three axioms stated above.

Example. An algebraic structure satisfying all laws of the algebra of sets is provided by a set of eight numbers, {1,2,3,5,6,10,15,30}, where addition is identified with obtaining the least common multiple, multiplication with the greatest common divisor, $m \subset n$ to mean "m is a factor of n", and $n' = 30/n$,

- $m + n \equiv LCM(n, m)$
- $m \cdot n \equiv GCD(n, m)$
- $m \subset n \equiv (n = 0 mod(m))$
- $n' \equiv 30/n$.

Exercise. Verify that thus obtained algebra satisfies all rules of set algebra.

We observe that laws of the algebra of sets look similar to the laws of propositional logic and predicate calculus, if we identify

$A \cap B = A \cdot B$, with conjunction (AND), A∧B

$A \cup B = A + B$, with disjunction (OR), A∨B

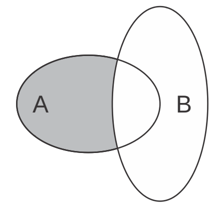
A' with negation (NOT), $\sim A$

$A \supset B$ with $A \Rightarrow B$, $A \subset B$ with $A \Leftarrow B$.

This is because any subset of a universal set can be defined using a predicate.

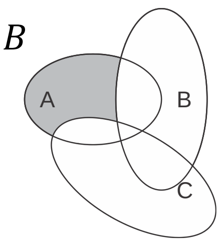
Definition. For two sets A, B, their difference $A - B$ (sometimes notation $A \backslash B$ is used instead of $A - B$) is defined by,

$$A - B \ = \ \{x: (x \in A) \wedge (x \notin B)\} = A \cap B'$$



The following properties can be shown to hold (consider Venn diagrams),

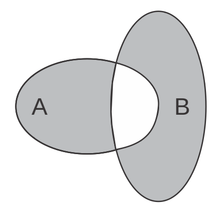
$A - (B \cup C) = \ (A - B) - C$, but in general, $A - (B - C) \neq (A \cup C) - B$



Definition. The symmetric difference of two sets is,

$$A \bigtriangleup B = (A - B) \cup (B - A)$$

This operation is commutative and associative,



$$A \bigtriangleup B = B \bigtriangleup A$$

$$(A \bigtriangleup B) \bigtriangleup C = A \bigtriangleup (B \bigtriangleup C)$$

Definition. For a set A, the characteristic function χ_A is defined as follows,

$$\chi_A(x) = \begin{cases} 1, if \ x \in A \\ 0, if \ x \ \notin A \end{cases}$$

Exercise. Show that χ_A has following properties

$$\chi_A = 1 - \ \chi_{A'}$$

$$\chi_{A \cap B} = \ \chi_A \, \chi_B$$

$$\chi_{A \cup B} = \ 1 - \ \chi_{A' \cap B'} = 1 - \ \chi_{A'} \, \chi_{B'} = 1 - (1 - \ \chi_A)(1 - \ \chi_B) = \chi_A + \chi_B - \chi_A \chi_B$$

Exercise. Write a formula for $\chi_{A \cup B \cup C}$; $\chi_{A \cup B \cup C \cup D}$.