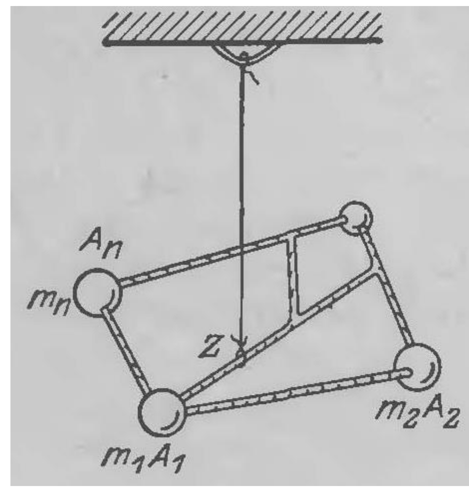
November 14, 2021

Geometry.

The Method of the Center of Mass (mass points): Solving problems using the Law of Lever (mass points). Menelaus theorem. Pappus theorem.

Heuristic Definitions of the Center of Mass (Center of Gravity) known to Greeks.

1. The point such that if suspended at it, an object will remain motionless in the equilibrium, independent of the position that it is placed.
2. The point common to all the lines passing through the point at which the object is suspended
3. The point common to all lines on which the object balances.



Archimedes' postulates on the properties of the Center of Gravity (COM).

1. The COM of similar figures are similarly situated.
2. The COM of a convex figure lies within the figure.
3. If an object is cut in two pieces, then its COM lies on the line joining the COM's of the pieces, and its position satisfies the Law of Lever.

However, the situation is much simpler if we only consider point masses.

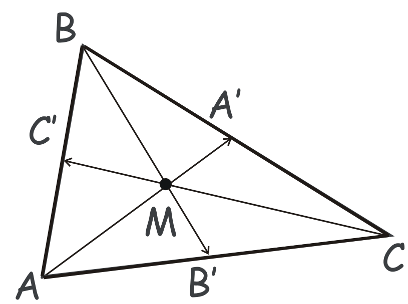
Properties of the Center of Mass for a system of point masses.

1. Every system of finite number of point masses has unique center of mass (COM).

2. For two point masses, m_1 and m_2, the COM belongs to the segment connecting these points; its position is determined by the Archimedes lever rule: the point's mass times the distance from it to the COM is the same for both points, $m_1 d_1 = m_2 d_2$.
3. The position of the system's center of mass does not change if we move any subset of point masses in the system to the center of mass of this subset. In other words, we can replace any number of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.

Solving problems using the COM.

Given a system of points and lines, one can derive various relations, such as concurrence of particular lines connecting some of the points, or the ratio of the lengths of different segments by associating certain masses with these points (i.e. placing point masses at their positions) and considering the center of mass of the obtained system of mass points.



Exercise. Prove that the medians of an arbitrary triangle ABC are concurrent (cross at the same point M).

Exercise. Prove that the bisectors of an arbitrary triangle ABC are concurrent (cross at the same point O).

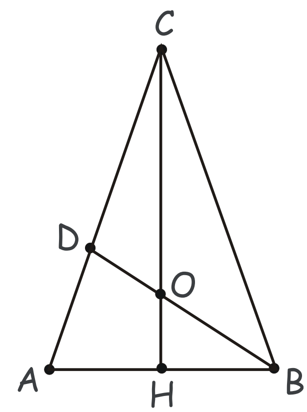
Exercise. Prove that the altitudes of an arbitrary triangle ABC are concurrent (cross at the same point O).

COM solutions of the selected homework problems.

1. **Problem.** Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).

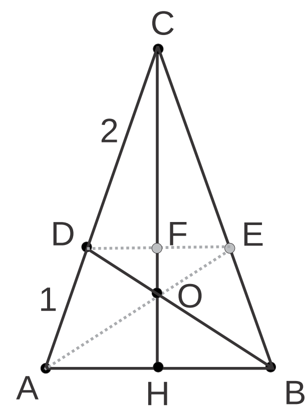
   Solution. Load vertices A, B and C with equal masses, m. Then, the center of mass (COM) of the three masses is at the intersection of the three medians, because it has to belong to each segment connecting the mass at the vertex of the triangle with the COM of the other two masses, i.e. the middle of the opposite side. COM this belongs to all three medians and is the centroid, O of the triangle. It divides each median in the 2:1 ratio because it is a COM of mass m at the vertex and a mass $2m$ at the middle of the opposite side.

2. **Problem**. In isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.



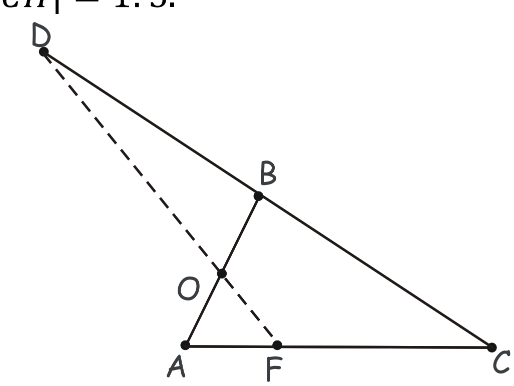
Solution.

a. Using the similarity and Thales theorem. First, let us perform a supplementary construction by drawing the segment DE parallel to AB, $DE||AB$, where point E belongs to the side CB, and point F to DE and the altitude CH. Notice the similar triangles, $AOH \sim DOF$, which implies, $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|}$. By Thales theorem, $\frac{|AH|}{|DF|} = \frac{|AC|}{|AD|} = 1 + \frac{|CD|}{|AD|} = \frac{3}{2}$, and $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|} = \frac{2}{3}$, so that $\frac{|FH|}{|OH|} = \frac{|FO|+|OH|}{|OH|} = \frac{5}{3}$. $\frac{|CH|}{|OH|} = \frac{|CH|}{|FH|}\frac{|FH|}{|OH|} = 3 \cdot \frac{5}{3} = 5$, because $\frac{|CH|}{|FH|} = 1 + \frac{|CF|}{|FH|} = 1 + \frac{|CD|}{|DA|}$. Therefore, the sought ratio is, $\frac{|OH|}{|CH|} = \frac{1}{5}$.



b. Using the Method of the Center of Mass. Load vertices A, B and C with masses $2m$, $2m$, and m, respectively. Then, H is the COM of masses at A and B, and D is the COM of masses at A and C, and O is the COM of all 3 masses in the vertices of the triangle ABC. Therefore, $|OC|:|OH| = (2m + 2m):m = 4:1$, $|OH|:|CH| = 1:5$.

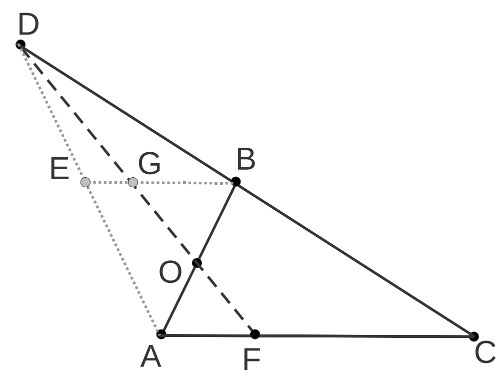
3. **Problem**. Point D belongs to the continuation of side CB of the triangle ABC such that $|BD| = |BC|$. Point F belongs to side AC,



and $|FC| = 3|AF|$. Segment DF intercepts side AB at point O. Find the ratio $|AO|:|OB|$.

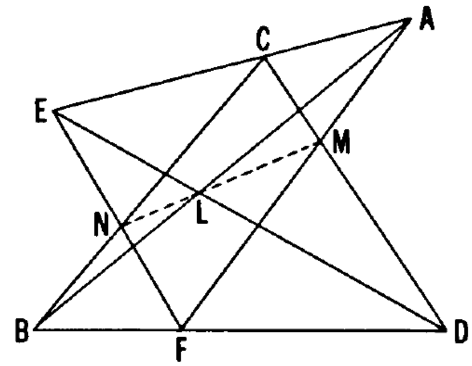
Solution.

a. Using the similarity and Thales theorem. First, let us perform a supplementary construction by drawing the segment BE parallel to AC, $BE||AC$, where E belongs to the side AD of the triangle ACD. BE is the mid-line of the triangle ACD, and, by Thales, also of AFD and FDC. Therefore, $|EG| = \frac{1}{2}|AF|$, $|GB| = \frac{1}{2}|FC|$ and $|EB| = \frac{1}{2}|AC|$, so $\frac{|BG|}{|EG|} = \frac{|FC|}{|AF|} = 3$. On the other hand, again, by Thales, or, noting similar triangles $AOF \sim BOG$, $\frac{|AO|}{|OB|} = \frac{|AF|}{|GB|} = 2\frac{|AF|}{|AC|} = \frac{2}{3}$.



b. Using the Method of the Center of Mass. Load vertices A, C and D with masses $3m$, m and m, respectively. Then, F is the center of mass (COM) of A and C, B is the COM of D and C, and O is the COM of the triangle ACD, $|AO|:|OB| = (m+m):3m = 2:3$.

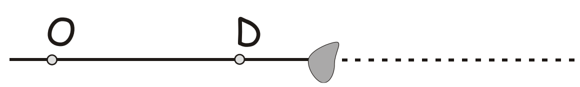
Theorem (Pappus). If A, C, E are three points on one line, B, D and F on another, and if three lines, AB, CD, EF, meet DE, FA, BC, respectively, then the three points of intersection, L, M, N, are collinear.



This is one of the most important theorems in planimetry, and plays important role in the foundations of projective geometry. There are a number of ways to prove it. For example, one can consider five triads of points, LDE, AMF, BCN, ACE and BDF, and apply Menelaus theorem to each triad. Then, appropriately dividing all 5 thus obtained equations, we can obtain the equation proving that LMN are collinear, too, also by the Menelaus theorem. However, one can prove the Pappus theorem directly, using the method of point masses.

Instead of simply proving the theorem, consider the following problem.

Problem. Using only pencil and straightedge, continue the line to the right of the drop of ink on the paper without touching the drop.



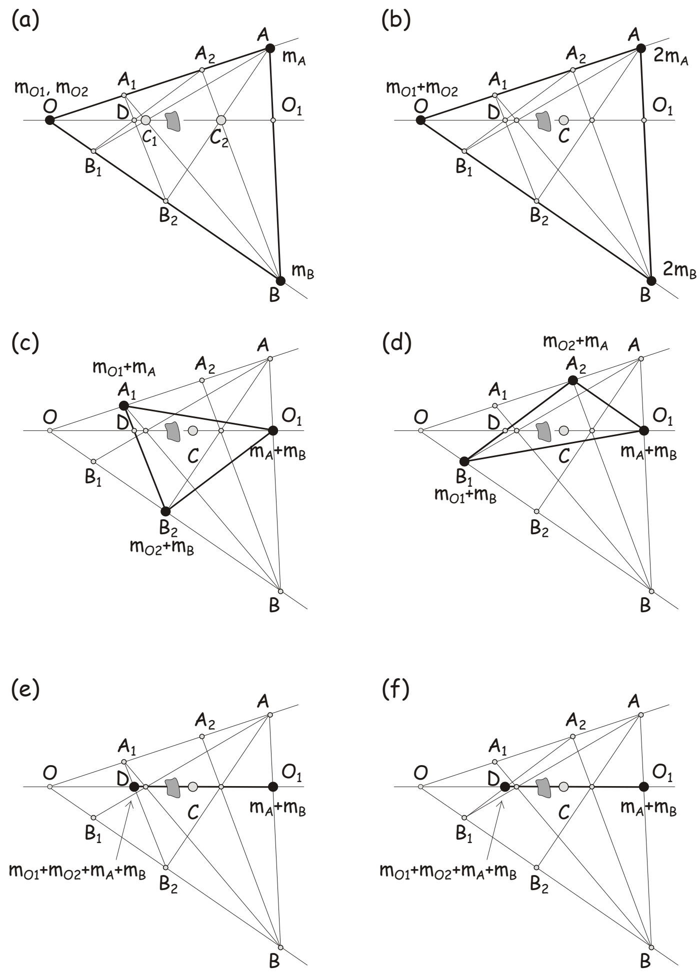
Solution by the Method of the Center of Mass.

Construct triangle OAB, which encloses the drop, and with the vertex O on the given line (OD). Let O_1 be the crossing point of (OD) and the side AB. Let us now load vertices A and B of the triangle with point masses m_A and m_B, such that their center of mass (COM) is at the point O_1. Then, each point of the (Cevian) segment OO_1 is the center of mass of the triangle OAB for some point mass m_O loaded on the vertex O. The (Cevian) segments from vertices A and B, which pass through the center of mass of the triangle C, connect each of these vertices with the center of mass of the other two vertices on the opposite side of the triangle, OB and OA, respectively.

For the mass m_{O1} loaded on the vertex O, the center of mass of the triangle is C_1, and the centers of mass of the sides OA and OB are A_1 and B_1, respectively.

Similarly, C_2, A_2 and B_2 are those for the mass m_{O2} on the vertex O. The COM of the side AB is always at the point O_1, independent of mass m_O.

If we can show that segments A_1B_2 and A_2B_1 cross the given line (OD) at the same point, D, then our problem is solved, as we can draw Cevians BA_2 and AB_2, whose crossing points are on the segment OO_1 on the other side of the drop, by sequentially drawing Cevians BA_1 and AB_1 and segments A_1B_2, B_1A_2, Figure 1(a).



Let us load vertices O, A and B with masses $m_{O1} + m_{O2}$, $2m_A$ and $2m_B$, respectively, Figure 1(b). The center of mass of OAB is now at some point C, in-between C_1 and C_2 (actually, it is not important where it is on the line OO_1). Let us now move point masses m_{O1} and m_A to their center of mass A_1 on the side OA, m_{O2} and m_B to their center of mass B_2 on the side OB, and m_A and m_B to their center of mass O_1 on the side AB. Now masses are at the vertices of the triangle $A_1B_2O_1$ with the same center of mass, C, Figure 1(c). Consequently, the crossing point D of segments A_1B_2 and OO_1 is the center of mass for masses $m_{O1} + m_A$ and $m_{O2} + m_B$ placed at points A_1 and B_2, respectively. Point C then is the center of mass for $m_{O1} + m_{O2} + m_A + m_B$ at point D and $m_A + m_B$ at point O_1, Figure 1(e). Repeating similar arguments for the triangle $A_2B_1O_1$, Figure 1(d,f), we see that point D is also the crossing point of segments A_1B_2 and OO_1. Therefore, D is the crossing point of all three segments, A_1B_2, A_2B_1 and OO_1, which completes the proof.