Homework for December 5, 2021.

Algebra.

Review the classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Rewrite the following properties of set algebra and partial ordering operations on sets in the form of logical propositions, following the first example.
   a. $[\mathrm{A} \cdot (\mathrm{B} + \mathrm{C}) = \mathrm{A} \cdot \mathrm{B} + \mathrm{A} \cdot \mathrm{C}] \Leftrightarrow \big[(\mathrm{x} \in \mathrm{A}) \wedge \big((\mathrm{x} \in \mathrm{B}) \vee (\mathrm{x} \in \mathrm{C})\big)\big] = \big[\big((\mathrm{x} \in \mathrm{A}) \wedge (\mathrm{x} \in \mathrm{B})\big) \vee \big((\mathrm{x} \in \mathrm{A}) \wedge (\mathrm{x} \in \mathrm{C})\big)\big]$
   b. $A + (B \cdot C) = (A + B) \cdot (A + C)$
   c. $(A \subset B) \Leftrightarrow A + B = B$
   d. $(A \subset B) \Leftrightarrow A \cdot B = A$
   e. $(A + B)' = A' \cdot B'$
   f. $(A \cdot B)' = A' + B'$
   g. $(A \subset B) \Leftrightarrow (B' \subset A')$
   h. $(A + B)' = A' \cdot B'$
   i. $(A' + B')' + (A' + B)' = A$
2. Using definitions from the classwork handout, devise logical arguments proving each of the following properties of algebra and partial ordering operations on sets and draw Venn diagrams where possible (hint: use problem #1).
   a. $\mathrm{A} \cdot (\mathrm{B} + \mathrm{C}) = \mathrm{A} \cdot \mathrm{B} + \mathrm{A} \cdot \mathrm{C}$
   b. $A + (B \cdot C) = (A + B) \cdot (A + C)$
   c. $(A \subset B) \Leftrightarrow A + B = B$
   d. $(A \subset B) \Leftrightarrow A \cdot B = A$
   e. $(A + B)' = A' \cdot B'$
   f. $(A \cdot B)' = A' + B'$
   g. $(A \subset B) \Leftrightarrow (B' \subset A')$
   h. $(A + B)' = A' \cdot B'$
   i. $(A' + B')' + (A' + B)' = A$
3. Verify that a set of eight numbers, {1,2,3,5,6,10,15,30}, where addition is identified with obtaining the least common multiple,

$$m + n \equiv LCM(n, m)$$

multiplication with the greatest common divisor,

$$m \cdot n \equiv GCD(n, m)$$

$m \subset n$ to mean "m is a factor of n",

$$m \subset n \equiv \left(n = 0 mod(m)\right)$$

and

$$n' \equiv 30/n$$

satisfies all laws of the set algebra.

4. For a set A, define the characteristic function χ_A as follows,

$$\chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x \notin A \end{cases}$$

Show that χ_A has following properties

$$\chi_A = 1 - \chi_{A'}$$
$$\chi_{A \cap B} = \chi_A \chi_B$$
$$\chi_{A \cup B} = 1 - \chi_{A' \cap B'} = 1 - \chi_{A'} \chi_{B'} = 1 - (1 - \chi_A)(1 - \chi_B)$$
$$= \chi_A + \chi_B - \chi_A \chi_B$$

Write formulas for $\chi_{A \cup B \cup C}, \chi_{A \cup B \cup C \cup D}$.

5. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

6. Using the continued fraction representation, find rational number, r, approximating $\sqrt{2}$ to the absolute accuracy of 0.0001.

7. Find x, where
   a. $$x = 1 + \{1,1,1,1, \dots\} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}$$
   b. $$x = \{1,2,2,2, \dots\} = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$$

8. Find the value of the continued fraction given by

$$x = 1 + \{1,2,3,3,3,\ldots\} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}}}$$

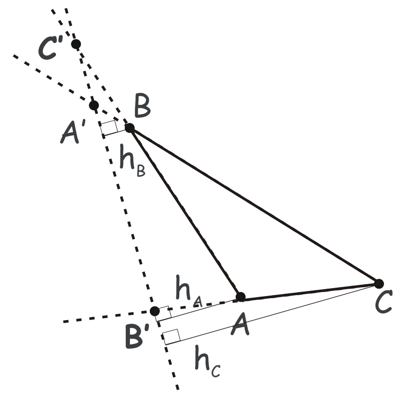
9. Consider the values of the following expression, y, for different x. How does it depend on x when n becomes larger and larger?

$$n\ fractions \left\{ \begin{array}{l} y = 3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \cdots}}} \\ \qquad\qquad\qquad \cdots - \cfrac{2}{3 - x} \end{array} \right. .$$

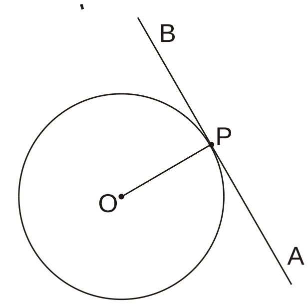
Geometry.

Review the last classwork handout on inscribed angles and quadrilaterals. Go over the proof of Ptolemy's theorem. Solve the unsolved problems from previous homework. Try solving the following problems.

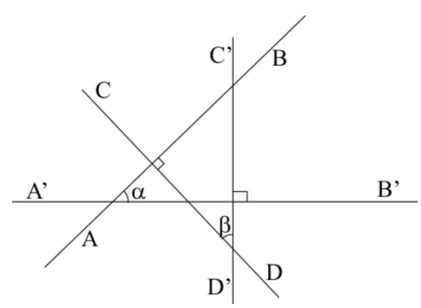
Problems.



1. Prove Menelaus theorem for the configuration shown on the right using mass points. Menelaus theorem states,
   Points C', A' and B', which belong to the lines containing the sides AB, BC and CA, respectively, of triangle ABC are collinear if and only if,
   $\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = 1$



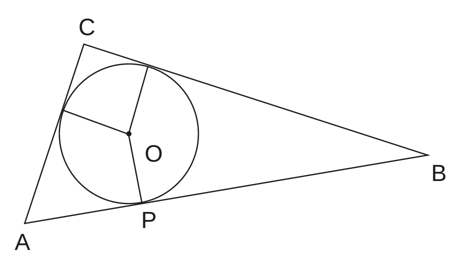
2. **Tangent line** to a circle is a line that has one and only one common point with the circle (definition). Prove that tangent line AB is perpendicular to the radius OP ending at the point P, which is the common point of the line and the circle (see Figure on the right).
3. We know from geometry that a circle can be drawn through the three vertices of any triangle. Find a radius of such circle if the sides of the triangle are 6, 8, and 10. (Gelfand and Saul "Trigonometry" p60, #4).



4. Prove that in the Figure on the right, $\angle\alpha$ is congruent to $\angle\beta$ if $AB \perp CD$ and $A'B' \perp C'D'$.
5. Using a compass and a ruler, draw a circle inscribed in the given triangle ABC. Prove the following formula for the area of the triangle,

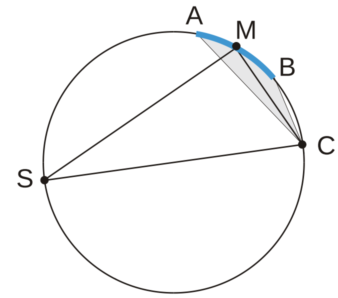
$$S_{ABC} = ½pr,$$

   where p is the perimeter of the triangle and r the radius of the inscribed circle.

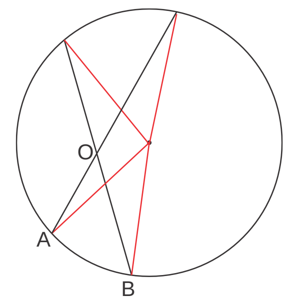


6. A **Rowland focusing** mirror is a device which focuses light of a certain color from the point source S onto a point, C, at sample. The mirror has the

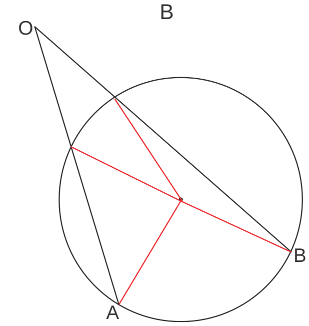
shape of a circular arc AB of 40 cm length. It is positioned so that its center, M, is at a distance of 4 m from the source S and at a distance 2 m from the sample C, $|SM| = 4$ m, $|MC| = 2$ m. The light ray of the color of interest is reflected so that it forms a 90˚ angle with the incident ray (e.g. angle SMC in the figure on the right is 90˚).



   a. What is the radius of the Rowland circle?
   b. What is the angular size of the light beam illuminating the sample (shaded angle ACB in the figure)? Does it depend on the position of sample, C?



7. Prove that an angle whose vertex lies inside a disk is measured by a semi-sum of the two arcs, one of which is intercepted by this angle, and the other by the angle vertical to it.



8. Prove that an angle whose vertex lies outside a disk and whose sides intersect the circle, is measured by a semi-difference the two intercepted arcs.
9. Given a circle and a diameter drawn of that circle, using only a straightedge, draw a perpendicular to that diameter passing through point P on the circle.