Homework for January 9, 2022.

Algebra.

Review the previous classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Construct bijections between the following sets:
   a. (subsets of the set $\{1,\dots,n\}$) $\leftrightarrow$ (sequences of zeros and ones of length n)
   b. (5-element subsets of $\{1,\dots,15\}$)$\leftrightarrow$ (10-element subsets of $\{1,\dots,15\}$)
   c. [set of all ways to put 10 books on two shelves (order on each shelf matters)]$\leftrightarrow$ (set of all ways of writing numbers 1, 2, . . . , 11 in some order) [Hint: use numbers 1. . . 10 for books and 11 to indicate where one shelf ends and the other begins.]
   d. (all integer numbers) $\leftrightarrow$ (all even integer numbers)
   e. (all positive integer numbers) $\leftrightarrow$ (all integer numbers)
   f. (interval (0,1))$\leftrightarrow$ (interval (0,5))
   g. (interval (0,1))$\leftrightarrow$ (halfline $(1,\infty)$) [Hint: try $1/x$.]
   h. (interval (0,1))$\leftrightarrow$ (halfline $(0,\infty)$)
   i. (all positive integer numbers)$\leftrightarrow$ (all integer numbers)
2. Let A be a finite set, with 10 elements. How many bijections f: $A \to A$ are there? What if A has n elements?
3. Let $f\colon \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ be given by $f(n) = n^2$. Is this function injective? surjective?
4. Hotel Infinity is a fictional hotel with infinitely many rooms, numbered 1, 2, 3, …. Each hotel room is single occupancy: only one guest can stay there at any time.
   a. At some moment, Hotel Infinity is full: all rooms are occupied. Yet, when 2 more guests arrive, the hotel manager says he can give rooms to them, by moving some of the current guests around. Can you show how? (Hint: Construct a bijection between the set $\{-1,0,1,2,\dots\}$ and the set of natural numbers, $\mathbb{N}$).
   b. At some moment, Hotel Infinity is full: all rooms are occupied. Still,

the management decides to close half of the rooms — all rooms with odd numbers — for renovation. They claim they can house all their guests in the remaining rooms. Can you show how? (Hint: Construct a bijection between the set of all even positive integers $\{2,4,6,\ldots\}$ and $\mathbb{N}$).

c. Next to Hotel infinity, a competitor has built Hotel Infinity 2, with infinitely many rooms numbered by all integers: $\ldots,-2,-1,0,1,2,\ldots$. Yet, the management of the original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 2 in Hotel Infinity. Could you show how? (Hint: Construct a bijection between the set of all integer numbers $\{\ldots,-2,$ $-1,0,1,2,\ldots\}$ and $\mathbb{N}$).

Bonus recap problems

5. Find the value of the continued fraction given by

$$\{1,2,3,3,3,\ldots\} = 1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cdots}}}}$$

6. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.
7. Using the recurrence relation obtained by solving the old hats problem, derive the formula for a derangement probability,

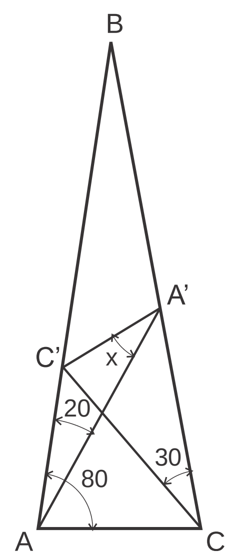
$$p_n \equiv \frac{!\,n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

8. Using the inclusion-exclusion principle, find how many natural numbers $n < 1000$ are divisible by 5, 7, 11, or 13.
9. How many passwords of at least 8 characters can one compose using lower- and upper-case letters and numbers 0 to 9?
10. * If 9 dies are rolled, what is the probability that all 6 numbers appear?
11. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?

Geometry.

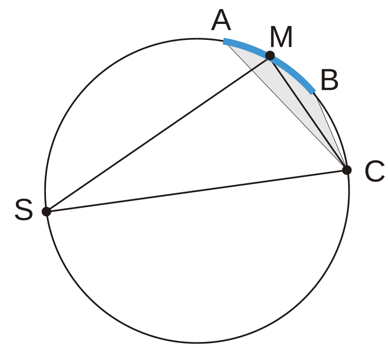
Review the last classwork handout on inscribed angles and quadrilaterals. Go over the proofs of Ptolemy's and Euclid's theorems. Try solving the following problems including the unsolved problems from previous homeworks.

Problems.



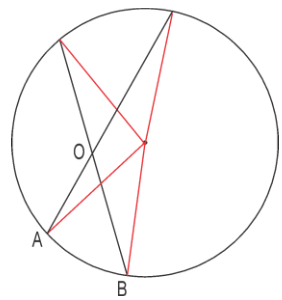
1. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.
2. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.
   a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?
   b. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?
   c. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle?
3. Consider all triangles with a given base and given altitude corresponding to this base. Prove that among all these triangles the isosceles triangle has the biggest angle opposite to the base.
4. Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.

5. A **Rowland focusing** mirror is a device which focuses light of a certain color from the point source S onto a point, C, at sample. The mirror has the shape of a circular arc AB of 40 cm length. It is positioned so that its center, M, is at a distance of 4 m from the source S and at a distance 2 m from the sample C, $|SM| = 4$ m, $|MC| = 2$ m. The light ray of the color of interest is reflected so that it forms a 90° angle with the incident ray (e.g. angle SMC in the figure on the right is 90°).

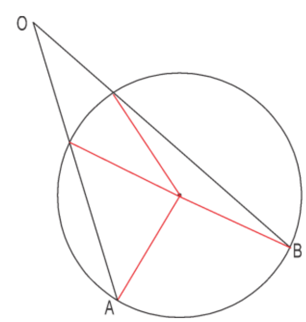


   a. What is the radius of the Rowland circle?
   b. What is the angular size of the light beam illuminating the sample (shaded angle ACB in the figure)? Does it depend on the position of sample, C?

6. Prove that an angle whose vertex lies inside a disk is measured by a semi-sum of the two arcs, one of which is intercepted by this angle, and the other by the angle vertical to it.



7. Prove that an angle whose vertex lies outside a disk and whose sides intersect the circle, is measured by a semi-difference the two intercepted arcs.



8. A point P is chosen on the chord AB of a circle with the center O and radius R. Show that $|AP| \times |PB| = R^2 - |OP|^2$.