Homework for January 16, 2022.

Algebra.

Review the classwork handout. Review and solve the classwork exercises which were not solved (some are repeated below). Solve the following problems (skip the ones you already solved).

1. Present examples of binary relations that are, and that are not equivalence relations.
2. For each of the following relations, check whether it is an equivalence relation and describe all equivalence classes.
   a. On $\mathbb{R}$: relation given by $x \sim y$ if $|x|=|y|$
   b. On $\mathbb{Z}$: relation given by $a \sim b$ if $a \equiv b \ mod\ 5$
   c. On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$; describe the equivalence class of $(1, 2)$
   d. Let $\sim$ be the relation on the set of all directed segments in the plane defined by $\overrightarrow{A\,B} \sim \overrightarrow{A'\,B'}$ if $ABB'A'$ is a parallelogram.
   e. On the set of pairs of integers, $\{(a, b), a, b \in \mathbb{Z}, b \neq 0\}$, $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 b_2 = a_2 b_1$. Describe these equivalence classes. Is the set of the obtained equivalence classes countable?
3. Let $f: X \xrightarrow{f} Y$ be a function. Define a relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Prove that it is an equivalence relation. Describe the equivalence classes for the equivalences defined by the following functions on $\mathbb{R}$.
   a. $f(x) = x^2$: $x \sim y$ if $x^2 = y^2$.
   b. $f(x) = \sin x$: $x \sim y$ if $\sin x = \sin y$.
4. Find the following sum. What is the smallest value of this sum for $x \in \mathbb{R}$?

$$\left(x - \frac{1}{x}\right)^2 + \left(x^2 - \frac{1}{x^2}\right)^2 + \cdots + \left(x^n - \frac{1}{x^n}\right)^2$$

5. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2? What is this ratio? What can you say about this triangle?
6. Solve the following equation,

$\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where x is a positive integer.

7. Find the following sum,
    a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
    b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n - 1) \cdot 3^n$
8. What is the minimum value of the expression, $(1 + x)^{36} + (1 - x)^{36}$ in the interval $|x| \leq 1$?

Geometry.

Review the previous classwork notes on the method of coordinates. No new geometry problems: please try solving the unsolved problems from the last homework, which are repeated below.

Problems.

1. Review the solution of the radical axis of two circles problem: find the locus of points whose powers with respect to two non-concentric circles are equal. Consider situation when circles are concentric.
2. Complete the following exercises from class. Find the locus of points satisfying each of the following equations or inequalities (graph it on a coordinate plane).
    a. $|x| = |y|$
    b. $|x| + x = |y| + y$
    c. $|x|/x = |y|/y$
    d. $[y] = [x]$
    e. $\{y\} = \{x\}$
    f. $x^2 - y^2 \geq 0$
    g. $x^2 + y^2 \leq 1$
    h. $x^2 + 8x = 9 - y^2$

3. Describe the locus of all points (x, y) equidistant to the X-axis (i. e. the line $y = 0$) and a given point P $(0,2)$ on the Y-axis. Write the formula relating y and x for these points.
4. (Skanavi 15.105) Find the (x, y) coordinates of the vertex C of an equilateral triangle ABC if A and B have coordinates $A(1,3)$ and $B(3,1)$, respectively.
5. (Skanavi 15.106) Find the (x, y) coordinates of the vertices C and D of a square $ABCD$ if A and B have coordinates $A(2,1)$ and $B(4,0)$, respectively.
6. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
7. **Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle.