Homework for January 23, 2022.

Algebra.

Review the classwork handout. Review the classwork and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Construct a proof that the set of all real numbers, $\mathbb{R}$, is uncountable, using the binary notation for real numbers.
2. Show that for the set of natural numbers, $\mathbb{N}$, cardinality of the set of all possible subsets is equal to that of a continuum of real numbers (Hint: use the binary number system).
3. Show that the set of points on any segment, $[a, b]$, on a line, has the same cardinality as
   a. the set of points on any other segment, $[c, d]$
   b. the set of points on a circle of unit radius
   c. the set of all points on a plane
   d. the set of all points in an n −dimensional hyper-cube
4. Show that each of the following sets has the same cardinality as a closed interval [0; 1] (i.e., there exists a bijection between each of these sets and [0; 1]).
   a. Interval [0; 1) [Hint: interval [0; 1] can be written as a union of two subsets, $A \cup B$, where A is a countable set including the interval end(s)].
   b. Open interval (0; 1)
   c. Set of all infinite sequences of 0s and 1s
   d. $\mathbb{R}$
   e. $[0,1] \times [0,1]$
5. Prove the following properties of countable sets. For any two countable sets, A, B,
   a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$

c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$
d. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the Cartesian product is also countable, $c(A_1 \times A_2 \ldots \times A_n) = \aleph_0$

6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. 1/15
   b. 1/14
   c. 1/7
   d. 1/6
   e. $0.33333\ldots = 0.(3)$
   f. $0.13333\ldots = 0.1(3)$

Geometry.

Review the previous classwork notes. Solve the remaining problems from the previous homework (some are repeated below). Solve the following problems.

Problems.

1. Consider all possible configurations of the Apollonius problem (i.e. different possible choices of circles, points and lines). How many possibilities are there? Make the corresponding drawings and write the equations for finding the Apollonius circle in one of them (of your choice).
2. (Skanavi 15.104) Given the right triangle ABC ($\angle C = 90°$), find the locus of points $P(x, y)$ such that $|PA|^2 + |PB|^2 = 2|PC|^2$.
3. (Skanavi 15.109) Points $A(-1,2)$ and $B(4,-2)$ are vertices of the rhombus $ABCD$, while point $M(-2,0)$ belongs to the side CD. Find the coordinates of the vertices C and D.
4. (Skanavi 15.114) Find the circle (write the equation of this circle) passing through the coordinate origin, $O(0,0)$, point $A(1,0)$ and tangent to the circle $x^2 + y^2 = 9$.
5. (Skanavi 15.115) Write the equation of the circle passing through the point $A(2,1)$ and tangent to both X- and Y-axes.
6. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.
7. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.