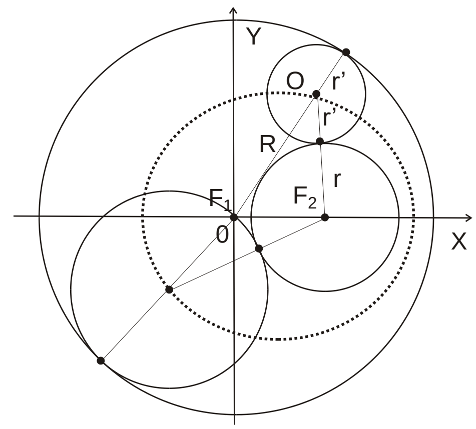
January 23, 2022 Math 9

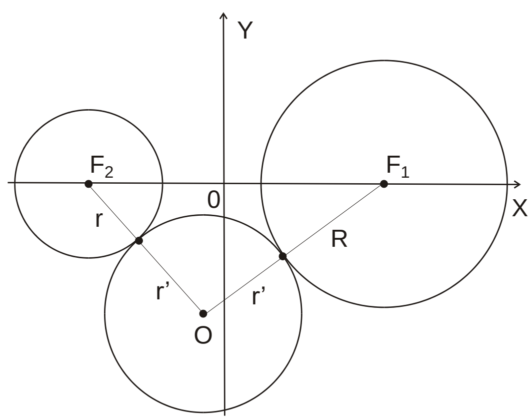
Geometry.

Ellipse. Hyperbola. Parabola (continued).

Alternate definitions of ellipse, hyperbola and parabola: Tangent circles.



Ellipse is the locus of centers of all circles tangent to two given nested circles (F_1, R) and (F_2, r). Its foci are the centers of these given circles, F_1 and F_2, and the major axis equals the sum of the radii of the two circles, $2a = R + r$ (if circles are externally tangential to both given circles, as shown in the figure), or the difference of their radii (if circles contain smaller circle (F_2, r).).



Consider circles (F_1, R) and (F_2, r). that are not nested. Then the loci of the centers O of circles externally tangent to these two satisfy $|OF_1| - |O\,F_2| = R - r$.

Hyperbola is the locus of the centers of circles tangent to two given non-nested circles. Its foci are the centers of these given circles, and the vertex distance $2a$ equals the difference in radii of the two circles.

As a special case, one given circle may be a point located at one focus; since a point may be considered as a circle of zero radius, the other given circle—which is centered on the other focus—must have radius $2a$. This provides a simple technique for constructing a hyperbola. It follows from this definition that a tangent line to the hyperbola at a point P bisects the angle formed with the two foci, i.e., the angle F_1PF_2. Consequently, the feet of perpendiculars drawn from each focus to such a tangent line lies on a circle of radius a that is centered on the hyperbola's own center.

If the radius of one of the given circles is zero, then it shrinks to a point, and if the radius of the other given circle becomes infinitely large, then the "circle" becomes just a straight line.

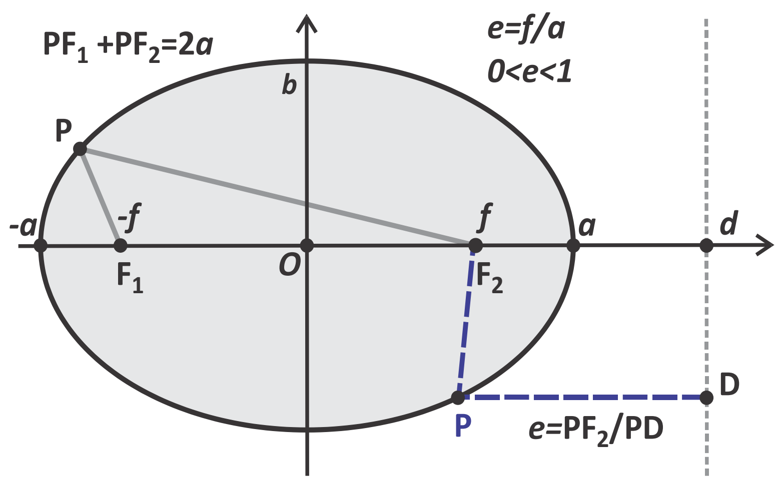
__Parabola__ is the locus of the centers of circles passing through a given point and tangent to a given line. The point is the focus of the parabola, and the line is the directrix.

Alternate definitions of ellipse, hyperbola and parabola: Directrix and Focus.

__Parabola__ is the locus of points such that the ratio of the distance to a given point (focus) and a given line (directrix) equals 1.

__Ellipse__ can be defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.

__Hyperbola__ can also be defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$. This constant is the eccentricity of the hyperbola. By symmetry a hyperbola has two directrices, which are parallel to the conjugate axis and are between it and the tangent to the hyperbola at a vertex.



In order to show that the above definitions indeed those of an ellipse and a hyperbola, let us obtain relation between the x and y coordinates of a point P (x, y) satisfying the definition. Using axes shown in the Figure, with focus F_2 on the X axis at a distance l from the origin and choosing the Y-axis for the directrix, we have

$$\frac{\sqrt{(x-l)^2 + y^2}}{x} = e$$

$$(x-l)^2 + y^2 = (ex)^2$$

$$x^2(1-e^2)\ -2lx+l^2+y^2=0$$

$$(1-e^2)\left(x^2-2x\frac{l}{1-e^2}+\left(\frac{l}{1-e^2}\right)^2\right)+y^2=\frac{l^2}{1-e^2}-l^2=\frac{e^2l^2}{1-e^2}$$

Finally, we thus obtain,

$$\frac{(x-\frac{l}{1-e^2})^2}{\frac{e^2l^2}{(1-e^2)^2}}+\frac{y^2}{\frac{e^2l^2}{1-e^2}}=1$$

Which is the equation of an ellipse for $1-e^2>0$ and of a hyperbola for $1-e^2<0$. In each case the center is at $x=x_0=\frac{l}{1-e^2}$ and $y=y_0=0$, and the semi-axes are $a=\frac{e\ l}{(1-e^2)}$ and $b=\frac{e\ l}{\sqrt{|1-e^2|}}$, which brings the equation to a canonical form,

$$\frac{(x-x_0)^2}{a^2}\pm\frac{(y-y_0)^2}{b^2}=1$$

We also obtain the following relations between the eccentricity e and the ratio of the semi-axes, a/b: $\frac{b}{a}=\sqrt{|1-e^2|}$, or, $e=\sqrt{1\pm\left(\frac{b}{a}\right)^2}$, where plus and minus sign correspond to the case of a hyperbola and an ellipse, respectively.

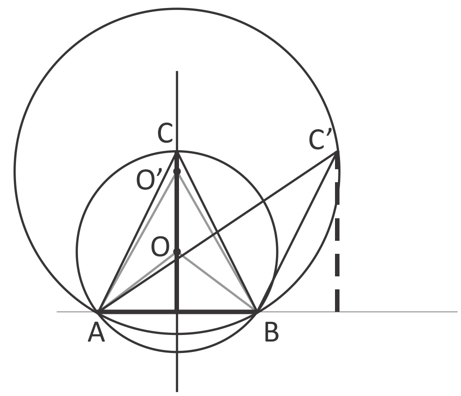
Curves of the second degree.

<u>A curve of the second degree</u> is a set of points whose coordinates in some (and therefore in any) Cartesian coordinate system satisfy a second order equation,

$$a_{11}x^2+a_{12}xy+a_{22}x^2+2b_1x+2b_2y+c=0$$

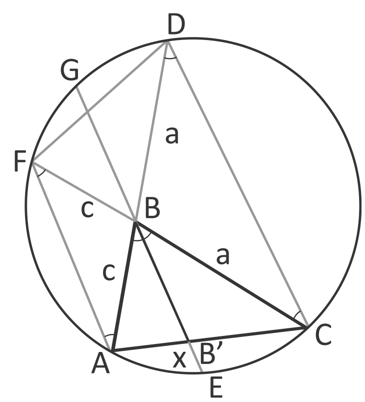
Solutions of some past homework problems.

1. **Problem**. Consider all triangles with a given base and given altitude corresponding to this base. Prove that among all these triangles the isosceles triangle has the biggest angle opposite to the base.



Solution. Consider a circumscribed circle for different triangles, an isosceles triangle ABC and some other triangle, ABC', which share the base AB and have the same altitude. For all such triangles, the center of the circumscribed circle will belong to the mid-perpendicular of the base AB, ie the altitude of an isosceles triangle on this base, or its continuation. If O is the center of the circle circumscribed around the isosceles triangle ABC and O' is the center of the circumscribed circle for any other triangle with the same altitude, ABC (on the same side of AB), then O' lies farther from AB than O (see figure). Consequently, $\angle AOB$ is larger than $\angle AO'B$. But by the inscribed angle theorem, $\angle AOB = 2\angle ACB$, $\angle AO'B = 2\angle AC'B$, and therefore, $\angle ACB > \angle AC'B$.

2. **Problem**. Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.



Solution. Consider the construction used to prove the property of a bisector: an isosceles triangle CBD, $CB = BD = a$. (Recap: the property of a bisector, BB', is obtained by applying Thales theorem to the angle DAC and two parallel lines, BB' and CD; we then obtain, $|AB'|:|B'C| = |AB|:|BC|$). Draw a circumscribed circle around the triangle ACD and extend the bisector BB to obtain the chord EG containing BB'. By symmetry, $|EB| = |BG|$ (see Figure). By the property of intersecting chords (Euclid's theorem), we have, $|AB||BD| = |EB||BG| = |EB|^2 = (|BB'| + |B'E|)^2$, wherefrom, $|BB'|^2 = |AB||BD| - |B'E|(|B'E| + 2|BB'|)$. On the other hand, by the same theorem, $|B'E||B'G| = |B'E|(|B'E| + 2|BB'|) = |AB'||B'C|$. Combining these two expressions, we obtain $|BB'|^2 = |AB||BC| - |AB'||B'C|$.

3. **Problem**. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.

Solution. Consider the figure. Find isosceles and congruent triangles (eg $|C'D| = |C'O|$, $|AC'| = |AC| = |AO|$, $\Delta A'C'D \cong \Delta A'C'O$, ...). It then follows that $\angle DC'O = \angle C'OA' = 100°$, and $x = 30°$.