Homework for February 13, 2022.

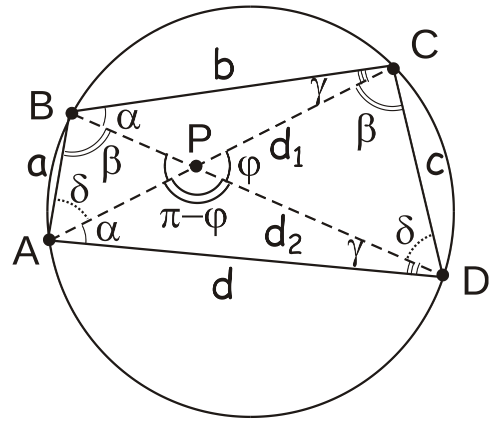
Geometry.

Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework (some are repeated below – skip the ones you have already done).

Problems.

1. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.
2. Find angle $\widehat{ACB}$ in the triangle ABC, given the lengths of the two adjacent sides, $|BC| = a$ and $|AC| = b$, and the length of the bisector of this angle, $|CC'| = l$.
3. Arc AB in a circle with the center O measures angle α. Line (BC) passes through point B, and the midpoint, C, of the radius OA. What is the ratio of the areas of the two parts into which this line divides sector AOB of the circle.
4. In an acute triangle ABC, the lengths of the two altitudes are $|AA'| = a$, and $|BB'| = b$, and the acute angle between these altitudes [lines (AA') and (AA')] is α. Find the length of the side $|AC|$.
5. Using the expressions for the sine and the cosine of the sum of two angles derived in class, derive expressions for (classwork exercise),
    a. $\sin 3\alpha$
    b. $\cos 3\alpha$
    c. $\tan(\alpha \pm \beta)$
    d. $\cot(\alpha \pm \beta)$
    e. $\tan(2\alpha)$
    f. $\cot(2\alpha)$
6. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.

7. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity (see Figure):



$$\sin(\alpha+\beta)\sin(\beta+\gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta,$$

if $\alpha+\beta+\gamma+\delta=\pi$.

8. Prove the Ptolemy identity in Problem 7 using the addition formulas for sine and cosine.
9. Using the Sine and the Cosine theorems, prove the Heron's formula for the area of a triangle,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s=\frac{a+b+c}{2}$ is the semi-perimeter.

Algebra.

Review the classwork handout and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Prove that the following numbers are irrational:
   a. $\sqrt[2]{2}$
   b. $\sqrt[3]{3}$
   c. $\sqrt[5]{5}$
2. Compare the following real numbers (are they equal? which is larger?)
   a. $\sqrt[3]{2}$ and $\sqrt[4]{3}$
   b. $\sqrt[4]{4}$ and $\sqrt[5]{5}$
   c. $\sqrt[10]{100}$ and $\sqrt[11]{101}$
   d. $\sqrt[100]{100}$ and $\sqrt[101]{101}$
3. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\, a, b \in \mathbb{R}$, one and only one of the following relations holds
    - $a = b$
    - $a < b$
    - $a > b$
2. $\forall\, a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

- $\forall\, a, b \in \mathbb{R}, a + b = b + a$
- $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\, a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\, a \in \mathbb{R}, \exists -a \in \mathbb{R}, a + (-a) = 0$
- $\forall\, a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\, a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$