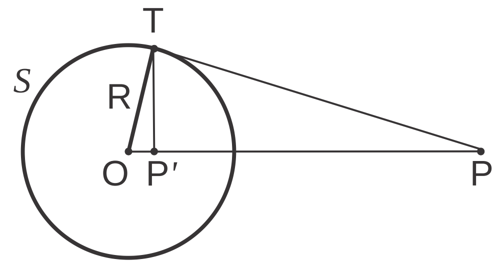
Geometry.

Inversive geometry.

Inversive geometry studies how properties of geometric figures change under the transformation of inversion. The inversion maps circles and lines into circles and lines, or, if we identify a line with a generalized circle of infinite radius, maps circles into circles, and preserves certain angles. Many difficult problems in geometry can be solved by applying an inversion transformation. For more details, see Wikipedia article on [inversive geometry], [cut-the-knot website], [mathworld website] and [mathematica article], and [Coxeter's article].

<u>Definition</u> (inversion of a plane). Given a circle S with the center at point O and radius R, inversion in the circle S is the transformation of the plane taking each point P to a point P' such that

i. the image P' belongs to a ray OP
ii. the distance $|OP'|$ satisfies relation, $|OP| \cdot |OP'| = R^2$.



<u>Remark</u>. Strictly speaking, the inversion is the transformation not of the entire plane but a plane without point O, the center of the inversion circle. The point O does not have an image (why?). It is convenient to think of an "infinitely remote" point O' added to a Euclidean plane so that O and O' are mapped to each other by the inversion.

"Inverting" a point in the plane with respect to a reference circle in geometry is a concept somewhat similar to obtaining the inverse of a number in algebra. Applied to all points of the Euclidean plane, circle inversion, or simply inversion, is a geometric transformation of the plane, which maps points lying inside the reference circle with center O and radius R onto points outside this circle, and vice versa. One can think of an inversion as of a reflection with respect to a circle, which is somewhat analogous to a reflection with respect to a straight line. If P' is the image of point P under inversion, the inverse of point P' under the same inversion is its pre-image, P. In other words, under inversion mapping the image and the pre-image of point P coincide: the inversion transformation applied twice is an identity.

Basic properties of inversion.

Consider inversion with respect to the reference circle S with center O and radius R. The observations below are almost obvious and easy to prove.

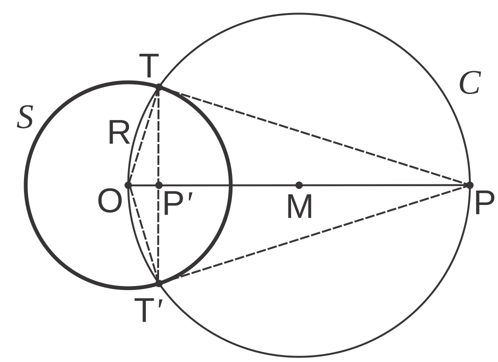
Observation 1. If $P \in S$, then $P' = P$. If P is inside S, then P' is outside S. If P is outside S, then P' is inside S.

Observation 2. If figure F under inversion goes into figure F', then F' under the same inversion goes into F. Another way to write this is $(F')' = F$.

Observation 3. If l is a straight line passing through the center of inversion ($O \in l$), then $l' = l$.

Observation 4 (compass and straightedge construction). To construct the inverse, P', of a point P outside an inversion circle S:

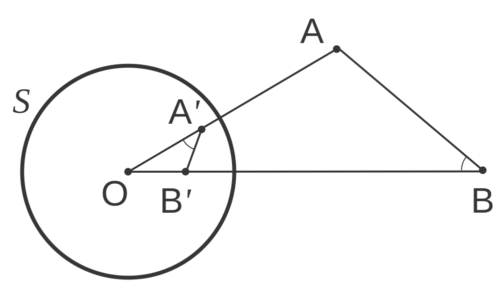
- Draw the segment OP from the center of the circle S to P.
- Let M be the midpoint of OP (recall how to plot a midpoint).
- Draw the circle C with center M going through P and O.
- Let T and T' be the points where S and C intersect.
- Draw segment TT'.
- P' is the intersection point of segments OP and TT'.



Exercise. Prove that P' is the inversion of point P in circle S. (Hint: consider similar right triangles POT and TOP').

Theorem. If points A' and B' are the images of the points A and B, respectively, under the inversion with the center O, then triangles OAB and $OB'A'$ are similar.

Proof. Triangles ΔOAB and $\Delta OB'A'$ share angle $\angle OAB$. Additionally, from the definition of inversion we have, $|OA'||OA| = |OB'||OB| = R^2$.



It then follows that the ratios of the

corresponding sides are equal,

$$\frac{|OA'|}{|OB|} = \frac{|OB'|}{|OA|}$$

This proves that ΔOAB and $\Delta OB'A'$ are similar, $\Delta OAB \sim \Delta OB'A'$.

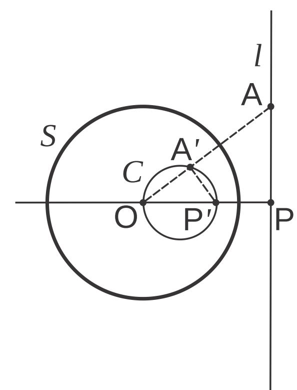
The statement of the above theorem is very useful in proving other properties of inversion, some of which are given as corollaries below.

<u>Corollary 1 (the distance formula)</u>. Using the similarity of ΔOAB and $\Delta OB'A'$ we obtain, $\frac{|A'B'|}{|AB|} = \frac{|OB'|}{|OA|} = \frac{|OA'|}{|OB|} = \frac{R^2}{|OA||OB|}$, or,

$$|A'B'| = \frac{R^2}{|OA||OB|}|AB|$$

<u>Corollary 2</u>. The straight line not passing through the center of inversion is mapped onto a circle passing through the center of inversion.

<u>Corollary 3</u>. The circle passing through the center of inversion is mapped onto a straight line not passing through the center of inversion.

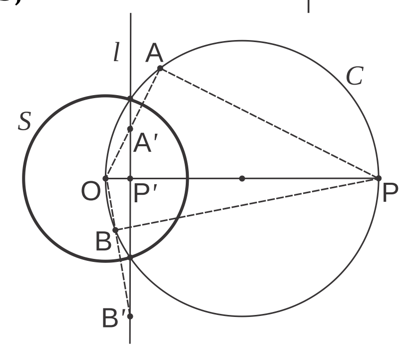


<u>Exercise</u>. Given line l and circle C, find the inversion circle, S, which transforms one into another. Consider three cases,

(i) l and circle C are crossing, i.e. have two common points

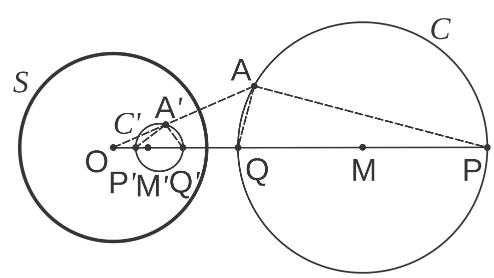
(ii) l and circle C are touching, i.e. have one common point

(iii) l and circle C have no common points



<u>Corollary 4</u>. The circle not passing through the center of inversion is mapped onto a circle not passing through the center of inversion.

<u>Remark</u>. Notice that the center of the image C' of a given circle C is NOT the image of the center of the circle C. Or in other words, while circles are



mapped into circles under the inversions, their centers are not.

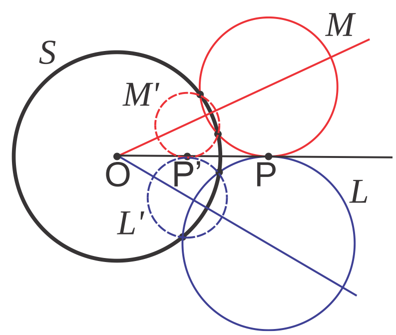
These corollaries are easy to prove by considering the similar triangles in the corresponding figures. In the case of Corollary 4, one should prove that if $P'Q'$ is an image of the diameter, PQ, of circle C and A' is an image of point A on that circle, then $\Delta P'A'Q'$ is a right triangle. The Corollaries 2-4 can be formulated as: “the lines and circles are mapped onto lines and circles under inversion”. In this context, it is convenient to think about lines as of circles of infinite radius. Then we can just say “circles are mapped onto circles”.

Exercise. Given circle C and its image C' of find the inversion circle, S, which transforms one into another. Consider three cases,

(i) circles C and C' are crossing, i.e. have two common points
(ii) circles C and C' are touching, i.e. have one common point
(iii) circles C and C' have no common points

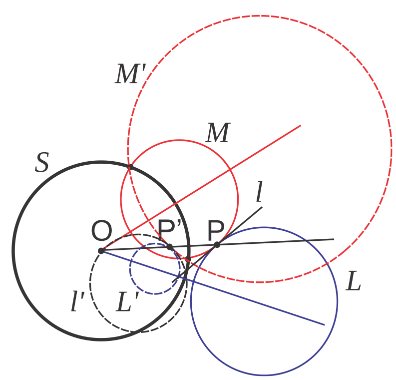
Tangency and conformal property of inversion.

As before, the following observations are nearly obvious and easy to prove (see figures), but are very useful.



Observation 5. If two circles, or circle and a straight line are tangent at point $P \neq O$, their images are also tangent, at point P' that is an image of point P (see figure).

Exercise. Prove this property (look at the figure and establish the correspondence between circles and their images).

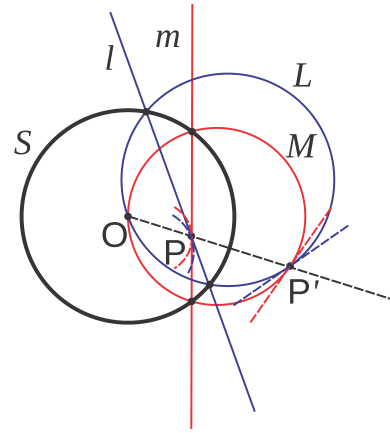


Observation 6. If two circles (or circle and straight line) are tangent at the center of inversion O, their images are two parallel straight lines.

Exercise. Find the distance between two parallel straight lines that are images of the two circles with the radii r_1 and r_2, which are tangent at the center O of the inversion circle S with radius R.

This observation shows how two parallel lines can be transformed into two circles, which is useful for solving geometrical problems.

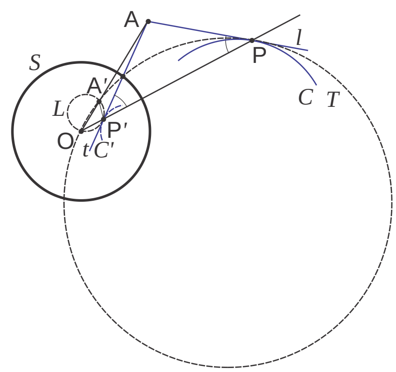
The following important theorem states that inversion "preserves angles" (such transformations are called conformal). The angle between arbitrary curved lines intersecting at point P is defined as the angle between tangents to those lines at the point P.



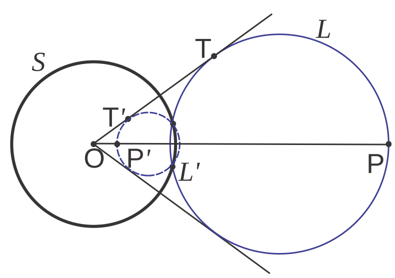
Theorem (conformal property). The angle of intersection of two curves, l and m, is equal to the angle of intersection of their images under inversion, l' and m'.

Proof. This property is an immediate consequence of the following lemma.

Lemma. If a curved line C intersects ray OP at point P at angle α, the image C' intersects ray OP at point P' at the same angle.



Proof. Consider the inversion of curve C in a circle S with the center O (see figure). Let l be tangent line to curve C at point P and circle L be the image of l. Let C' be the image of curve C and t be the tangent line to C' at point P', which is the image of point P. Then, t is also tangent to L and l is tangent to its pre-image, T, at points P' and P, respectively. Let A be the intersection point of the two tangent lines, l and t. The image A' is the intersection point of circles L and T and therefore OA is the common chord of these circles. It then follows that $|OA'|\cdot|OA| = |AP'|^2 = |AP|^2$. Hence, triangle APP' is isosceles and $\angle APO \cong \angle AP'P$, $\widehat{APO} = \widehat{AP'P}$.

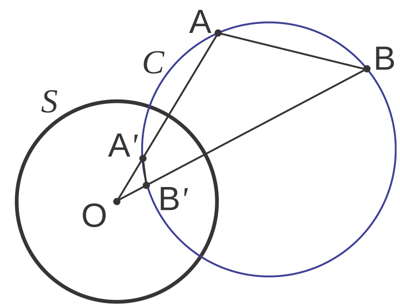


Observation 7. If circle L' is an image of circle L with respect to inversion with center O, L' and L are homothetic with respect to the center of inversion, O. However, the corresponding points, $P \in L$ and its inverse $P' \in L'$ are anti-homologous with respect to this homotethy, rather than homologous. If P is farther from the homotethy center, P' is closer and vice versa; the opposite is true for the homologous points in a homotethy. Common tangents of L and L' pass through O; the

inversion image of the tangency point $T \in L$ is the tangency point $T' \in L'$ on the same tangent line.

Exercise. Express the similarity coefficient between circle L and its image L' through radius of the inversion circle R and length of the tangent, $|OT|$. What happens if $|OT| = R$?

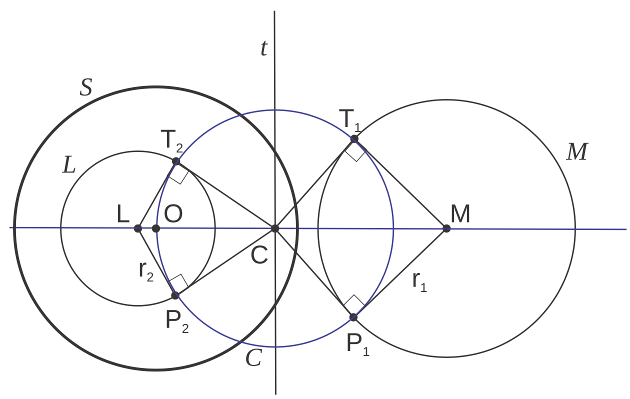
Observation 5. If points A' and B' are the images of points A and B, respectively, under the inversion with respect to circle S, then all four points are cyclic, i.e. lie on some circle C. In other words, $ABB'A'$ is an inscribed quadrilateral. This observation allows identifying circles that are invariant with respect to inversion.



Exercise. Prove this observation (hint: recall theorem on chords in a circle). Which circles are invariant with respect to the inversion in a circle S?

Mapping circles on concentric circles.

Theorem. For any pair of non-intersecting circles L and M (or a circle and a straight line), there exists an inversion that maps these circles onto a pair of concentric circles, L' and M'.



Proof. Consider point C of intersection of line LM connecting the centers of the two given circles and the radical axis of these circles, line t. By definition, all points on the radical axis have the same power with respect to given circles, L and M, and therefore the tangent segments from this point to both circles are equal, $|CT_1| = |CT_2| = |CP_1| = |CP_2| = R$. Here, T_1, P_1 and T_2, P_2 are tangency points on the first and the second circle, respectively (see figure). Circle with the center C and radius R passing through all four tangency points is perpendicular to both given circles and intersects line LM at point O. Consider inversion with respect to circle S with the center at this point O. Line LM will transform into itself, while circle C will transform into another line, because both pass through the center of inversion, while the two given circles will transform into another circles, L' and M'. By conformal property of inversion, the images L' and M' will be perpendicular to two intersecting straight lines, the images of line LM and circle C, because both line LM and circle C are perpendicular to given circles L and M. It then follows that circles L' and M' are concentric.

Ptolemy's inequality theorem by inversion.

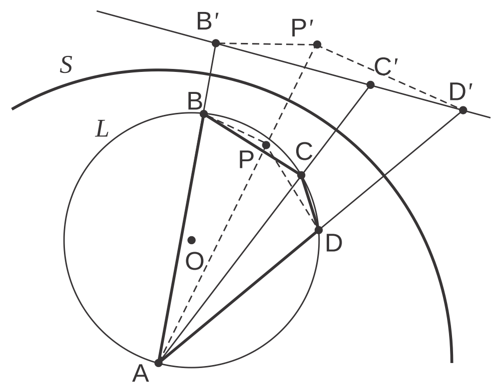
Ptolemy's inequality is an extension of Ptolemy's theorem for an inscribed quadrilateral.

Theorem. For any quadrilateral $ABCD$,

$$|AB| \cdot |CD| + |BC| \cdot |AD| \geq |AC| \cdot |BD|$$

where the equality is achieved when quadrilateral $ABCD$ is inscribed in a circle.

Proof. Let points A, B, C, and D be concyclic, i. e. quadrilateral $ABCD$ inscribed in a circle, L. Consider inversion with the center at the vertex of the quadrilateral, A, and radius R. It transforms cirlcle L into a line and the images of the three other vertices, points B', C', and D', lie on that line (see figure). It then follows that



$$|B'C'| + |C'D'| = |B'D'|$$

If point C is not on the circle, its image, C', is not on the line (cf points P and P' in the figure; it does not matter whether C is inside or outside the circle). Then, by triangle inequality,

$$|B'C'| + |C'D'| \geq |B'D'|$$

Using the distance formula this can be rewritten as,

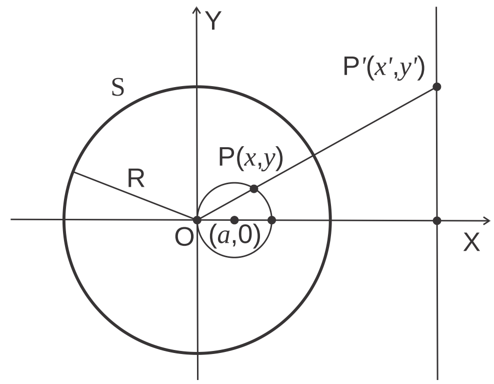
$$|BC|\frac{R^2}{|AB||AC|} + |CD|\frac{R^2}{|AC||AD|} \geq |BD|\frac{R^2}{|AB||AD|}$$

Or,

$$|BC| \cdot |AD| + |CD| \cdot |AB| \geq |AC| \cdot |BD|$$

Inversion in coordinate plane.

Consider inversion with respect to circle S centered at the origin, (0,0). Image of point $P(x,y)$ is point $P'(x',y')$. It is easy to see that the transformation of coordinates is (see figure),



$$x' = x\frac{R^2}{x^2+y^2}$$
$$y' = y\frac{R^2}{x^2+y^2}$$

And the inverse transformation,

$$x = x'\frac{R^2}{x'^2+y'^2}$$
$$y = y'\frac{R^2}{x'^2+y'^2}$$

The image of the line $y = ax$ is the line $y' = ax'$. Consider the image of the circle,

$$(x-a)^2 + y^2 = r^2$$

For points $P(x,y)$ on the circle, $x^2 + y^2 = r^2 - a^2 + 2ax$, so we have,

$$x' = x\frac{R^2}{r^2 - a^2 + 2ax}$$
$$y' = y\frac{R^2}{r^2 - a^2 + 2ax}$$

In the case where $a = r$, i.e. circle passes through the center of inversion, the image is a line,

$$x' = \frac{R^2}{2a}$$

<u>**Exercise**</u>. Show that in the case $a \neq r$ there exist x_0, y_0, r_0, such that the image of circle $(x-a)^2 + y^2 = r^2$ is circle $(x' - x_0)^2 + (y' - y_0)^2 = r_0^2$.