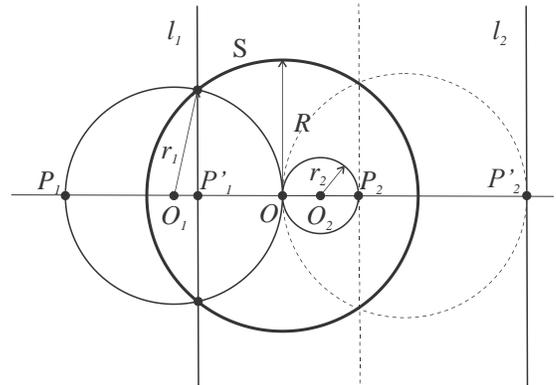# Geometry.

Inversive geometry. Homework review.

Problem. Find the distance between two parallel straight lines that are images of the two circles with the radii r_1 and r_2, which are tangent at the center O of the inversion circle S with radius R.


Solution. Consider the figure. Under the inversion in the circle (O, R), line l_1 is the image of the circle (O_1, r_1) and line l_2 is the image of the circle (O_2, r_2). The distance between the two lines, $|P'_1 P'_2| = |OP'_1| + |OP'_2| = \dfrac{R^2}{|OP_1|} + \dfrac{R^2}{|OP_2|} = \dfrac{R^2}{2r_1} + \dfrac{R^2}{2r_2}$.

Here, the pre-images of P'_1 and P'_2 are the points P_1 and P_2, which are intersections of the circles (O_1, r_1) and (O_2, r_2) with the line $O_1 O_2$ connecting their centers and also passing through the inversion center, O.

Problem. Consider a circle S with center O and a straight line PQ that cuts from S a circular segment PSQ.


   a. Prove that for any circle inscribed in the segment the line joining the tangency points A and B with the segment and with the circle passes through the midpoint M of the arc PMQ complementary to the segment.
   b. Prove that if two circles inscribed in a circular segment PSQ touch, their common tangent passes through M.
   c. Prove that if two circles inscribed in a circular segment PSQ cross, the line through the two points of intersection passes through M.
   d. A circle overlaps a circular segment so that the four angles it forms with the boundary of the segment are all equal. Let the points of
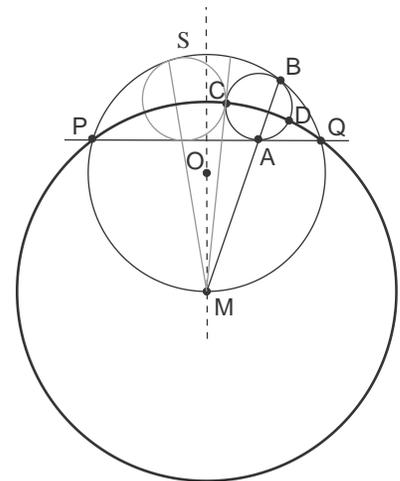
intersection be A_1 and A_2 on the linear segment and B_1 and B_2 on the arc such that A_1B_2 intersect A_2B_1 inside the segment. Then A_1B_1 and A_2B_2 meet in M.

e. A circle with center on PQ intersects PQ in A_1 and A_2 and S in B_1 and B_2 (A_1 is inside S, while B_1 is above PQ.) Prove that, if the two cricles meet at $90°$, then both A_1B_1 and A_2B_2 pass through M.

Solution. Consider the inversion in a circle with the center at the midpoint M of the arc PMQ complementary to the segment PSQ and passing through the points P and Q (see figure). Under such an inversion, line PQ is the image of the circle S, and vice versa. Hence, the tangency points A and B are the images of each other and therefore both belong to the line connecting either of them with the center of inversion, M. Furthermore, the intersection points, C and D, of a circle inscribed in a circular segment PSQ and the inversion circle with center M are invariant with respect of this inversion. Therefore, (i) MC and MD are tangent to the circle inscribed in the segment and (ii) all such inscribed circles are invariant with respect to this inversion. The above considerations substantiate (a) through (c). (d) and (e) are proven similarly, using the conformal property.

Problem. Steiner's Porism Theorem [Geometry Revisited, p. 124]. Given two circles - one inside the other. Pick up a point in-between and draw a circle tangent to the given two. Then draw a circle tangent to the new circle and the original two. Continue building a chain of circles each touching the two given circles and its predecessor in the chain. It may happen that, for some n, the n-th circle will touch the first circle in the chain. Prove that if this happens, it will happen regardless of the position of the starting point.
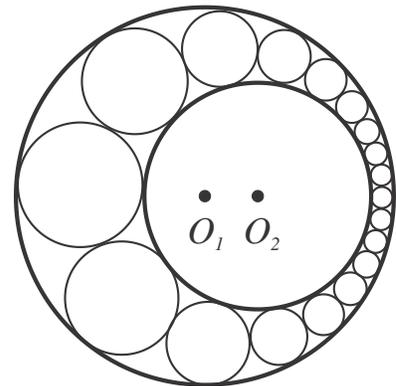

Solution. Consider the inversion that images the two nested circles in the problem onto the two concentric circles. Under this inversion, all nested tangent circles are imaged onto congruent circles tangent to the two concentric nested circles. The centers of these circles can be moved at will, and so can the centers of their pre-images in the figure.

Problem. Show that in the case $a \neq r$ there exist x_0, y_0, r_0, such that the image of circle $(x - a)^2 + y^2 = r^2$ is circle $(x' - x_0)^2 + (y' - y_0)^2 = r_0^2$.

<u>Solution</u>. Under an inversion with respect to a circle S centered at the origin, $(0,0)$, the image of point $P(x, y)$ is point $P'(x', y')$. The transformation of the coordinates is (see figure),

$$x' = x \frac{R^2}{x^2 + y^2}$$

$$y' = y \frac{R^2}{x^2 + y^2}$$
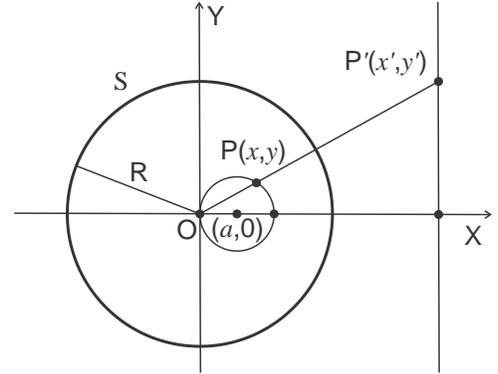
Hence,

$$x'^2 + y'^2 = \frac{R^4}{x^2 + y^2}$$

and

$$x = x' \frac{R^2}{x'^2 + y'^2}$$

$$y = y' \frac{R^2}{x'^2 + y'^2}$$

We thus have,

$$(x - a)^2 + y^2 = r^2 \Leftrightarrow x^2 + y^2 - 2ax + a^2 = r^2 \Leftrightarrow$$

$$\frac{R^4}{x'^2 + y'^2} - 2ax' \frac{R^2}{x'^2 + y'^2} = r^2 - a^2 \Leftrightarrow$$

$$R^4 - 2axR^2 = (x'^2 + y'^2)(r^2 - a^2) \Leftrightarrow$$

$$x'^2 + 2ax' \frac{R^2}{r^2 - a^2} + y'^2 = \frac{R^4}{r^2 - a^2} \Leftrightarrow$$

$$\left(x' + \frac{aR^2}{r^2 - a^2}\right)^2 + y'^2 = \frac{R^4}{r^2 - a^2} + a^2 \frac{R^4}{(r^2 - a^2)^2} = \frac{r^2 R^4}{(r^2 - a^2)^2}$$

Wherefrom we find, $x_0 = \frac{aR^2}{a^2 - r^2}$ and $r_0 = \left|\frac{rR^2}{r^2 - a^2}\right|$.