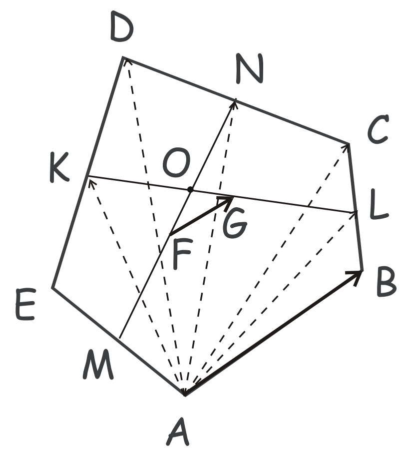
April 10, 2022 Math 9

Geometry.

Solving vector problems.

Problem. In a pentagon $ABCDE$, M, K, N and L are the midpoints of the sides AE, ED, DC, and CB, respectively. F and G are the midpoints of thus obtained segments MN and KL (see Figure). Show that the segment FG is parallel to AB and its length is ¼ of that of AB, $|FG| = 1/4|AB|$.



Solution. Express $\overrightarrow{FG}$ via sides of the pentagon,

$$\overrightarrow{FG} = \frac{1}{2}\overrightarrow{NM} + \frac{1}{2}\overrightarrow{EA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{LK},$$

$$\overrightarrow{NM} = \frac{1}{2}\overrightarrow{CD} + \overrightarrow{DE} + \frac{1}{2}\overrightarrow{EA},$$

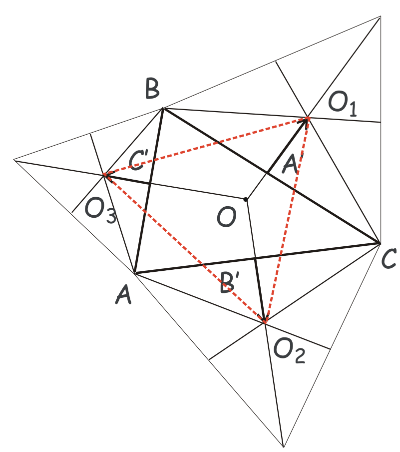
$$\overrightarrow{LK} = \frac{1}{2}\overrightarrow{BC} + \overrightarrow{CD} + \frac{1}{2}\overrightarrow{DE}.$$

$$\overrightarrow{FG} = \frac{1}{2}\left(\frac{1}{2}\overrightarrow{CD} + \overrightarrow{DE} + \frac{1}{2}\overrightarrow{EA}\right) + \frac{1}{2}\overrightarrow{EA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\left(\frac{1}{2}\overrightarrow{BC} + \overrightarrow{CD} + \frac{1}{2}\overrightarrow{DE}\right),\text{ or,}$$

$$\overrightarrow{FG} = \frac{3}{4}\overrightarrow{BC} + \frac{3}{4}\overrightarrow{CD} + \frac{3}{4}\overrightarrow{DE} + \frac{3}{4}\overrightarrow{EA} + \overrightarrow{AB} = \frac{3}{4}\left(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA}\right) + \frac{1}{4}\overrightarrow{AB}$$

Or, $\overrightarrow{FG} = \frac{1}{4}\overrightarrow{AB}$, since $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = 0$.

Problem. Three equilateral triangles are erected externally on the sides of an arbitrary triangle ABC. Show that triangle $O_1O_2O_3$ obtained by connecting the centers of these equilateral triangles is also an equilateral triangle (Napoleon's triangle, see Figure).



Solution. Denote $|AB| = c$, $|BC| = a$, $|AC| = b$. Let us find the side $|O_2O_3|$. Express $\overrightarrow{O_2O_3} = \overrightarrow{AO_3} - \overrightarrow{AO_2}$, or, $\overrightarrow{O_2O_3} = \frac{1}{2}\overrightarrow{AB} + \overrightarrow{C'O_3} - \frac{1}{2}\overrightarrow{AC} - \overrightarrow{B'O_2}$.

Note, that $\left|\overrightarrow{B'O_2}\right| = b\frac{\sqrt{3}}{6}$, and $\left|\overrightarrow{C'O_3}\right| = c\frac{\sqrt{3}}{6}$. Also, $\left(\overrightarrow{AB}\cdot\overrightarrow{AC}\right) = bc\cos\alpha$,
$\left(\overrightarrow{AB}\cdot\overrightarrow{B'O_2}\right) = \left(\overrightarrow{AC}\cdot\overrightarrow{C'O_3}\right) = bc\frac{\sqrt{3}}{6}\cos(90^\circ+\alpha) = -\frac{1}{2\sqrt{3}}bc\sin\alpha$, and
$\left(\overrightarrow{C'O_3}\cdot\overrightarrow{B'O_2}\right) = \frac{1}{12}bc\cos(180^\circ-\alpha) = -\frac{1}{12}bc\cos\alpha$, where $\alpha = \widehat{BAC}$. Then,

$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{4}\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{C'O_3}\right|^2 + \frac{1}{4}\left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{B'O_2}\right|^2 - \frac{1}{2}\left(\overrightarrow{AB}\cdot\overrightarrow{AC}\right)$$

$$-\left(\overrightarrow{AB}\cdot\overrightarrow{B'O_2}\right) - \left(\overrightarrow{AC}\cdot\overrightarrow{C'O_3}\right) - 2\left(\overrightarrow{C'O_3}\cdot\overrightarrow{B'O_2}\right),$$ or,

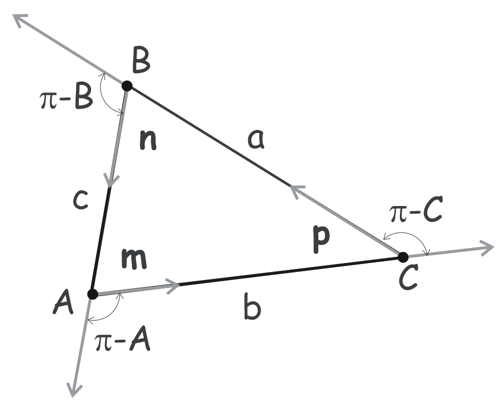
$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{4}\left(c^2 + \frac{1}{3}c^2 + b^2 + \frac{1}{3}b^2 - 2bc\cos\alpha + \frac{4}{\sqrt{3}}bc\sin\alpha + \frac{2}{3}bc\cos\alpha\right),$$

$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{3}c^2 + \frac{1}{3}b^2 - \frac{1}{3}bc\cos\alpha + \frac{1}{\sqrt{3}}bc\sin\alpha.$$

Now, using the Law of cosines, $2bc\cos\alpha = b^2 + c^2 - a^2$, and the Law of sines, $\sin\alpha = \frac{a}{2R}$, where R is the radius of the circumcircle, we obtain $\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{6}a^2 + \frac{1}{6}b^2 + \frac{1}{6}c^2 + \frac{abc}{2\sqrt{3}R}$. Obviously, the same expression holds for the sides $|O_1O_3|$ and $|O_1O_2|$. Hence, triangle $O_1O_2O_3$ is equilateral.

Problem. Let A, B and C be angles of a triangle ABC.

a. Prove that $\cos A + \cos B + \cos C \leq \frac{3}{2}$.
b. *Prove that for any three numbers, m,n,p, $2mn\cos A + 2np\cos B + 2pm\cos C \leq m^2 + n^2 + p^2$



Solution. Let vectors $\vec{m}, \vec{n}, \vec{p}$ be parallel to $\overrightarrow{AC}$, $\overrightarrow{BA}$ and $\overrightarrow{CB}$, respectively, as in the Figure. Then,

$$(\vec{m} + \vec{n} + \vec{p})^2 = m^2 + n^2 + p^2 - 2mn\cos A - 2np\cos B - 2mp\cos C$$

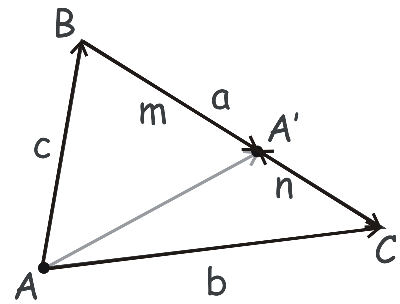
wherefrom immediately follows that,

$$2mn\cos A + 2np\cos B + 2pm\cos C \leq m^2 + n^2 + p^2.$$

The statement in part (a) follows from the above for $m = n = p = 1$.

Problem. Point A' divides the side BC of the triangle ABC into two segments, BA' and $A'C$, whose lengths have the ratio $|BA'|:|A'C| = m:n$. Express vector $\overrightarrow{AA'}$ via vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Find the length of the Cevian AA' if the sides of the triangle are $|AB| = c$, $|BC| = a$, and $|AC| = b$.

Solution. It is clear from the Figure, that $\overrightarrow{BA'} = \frac{m}{n}\overrightarrow{A'C} = \frac{m}{m+n}\overrightarrow{BC}$, and $\overrightarrow{CA'} = \frac{n}{m+n}\overrightarrow{CB} = \frac{n}{m+n}\left(\overrightarrow{AB} - \overrightarrow{AC}\right)$. Therefore,



$$\overrightarrow{AA'} = \overrightarrow{AC} + \overrightarrow{CA'} = \overrightarrow{AC} + \frac{n}{m+n}\left(\overrightarrow{AB} - \overrightarrow{AC}\right) = \frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}.$$

Or, we can obtain the same result as

$$\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{BA'} = \overrightarrow{AB} + \frac{m}{m+n}\left(\overrightarrow{AC} - \overrightarrow{AB}\right) = \frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}.$$

For the length of the segment AA' we have,

$|AA'|^2 = \overrightarrow{AA'}^2 = \left(\frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}\right)^2 = \frac{n^2c^2 + m^2b^2 + (nm)2bc\cos\widehat{BAC}}{(m+n)^2}$. Using the Law of cosines, we write $2bc\cos\widehat{BAC} = b^2 + c^2 - a^2$, and obtain the final result,

$$|AA'|^2 = \frac{(n^2+nm)c^2 + (m^2+nm)b^2 - (mn)a^2}{(m+n)^2} = \frac{mb^2+nc^2}{m+n} - \frac{mna^2}{(m+n)^2}.$$

Or, equivalently, $(m+n)|AA'|^2 = mb^2 + nc^2 - \frac{mna^2}{m+n}$.

Substituting $m + n = a$, we obtain the Stewart's theorem (Coxeter, Greitzer, exercise 4 on p. 6).

<u>If AA' is a median</u>, then $|BA'|:|A'C| = 1:1$, i.e. $m = n = 1$, and we have, $\overrightarrow{AA'} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$, $|AA'|^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$ (AA' is a median).

<u>If AA' is a bisector</u>, $|BA'|:|A'C| = c:b$, i.e. $m = c, n = b$, and we obtain $\overrightarrow{AA'} = \frac{b}{b+c}\overrightarrow{AB} + \frac{c}{b+c}\overrightarrow{AC}$, as well as $|AA'|^2 = \frac{b^2c+c^2b}{b+c} - \frac{bca^2}{(b+c)^2} = bc\left(1 - \frac{a^2}{(b+c)^2}\right)$ (AA' is a bisector).

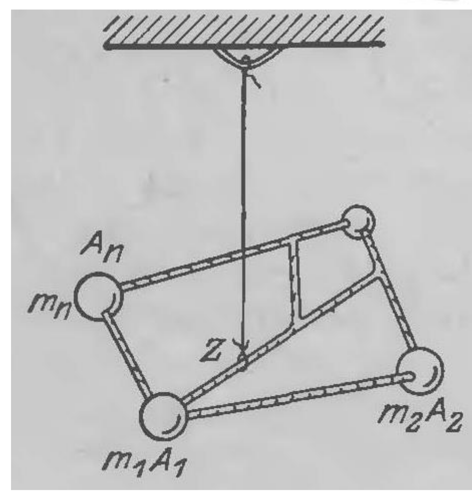
Recap: Vector definition of the center of mass.

Let us assume that a system of geometric points, $X_1, X_2, X_3, \dots, X_n$ has masses $m_1, m_2, m_3, \dots, m_n$ associated with each point. The total mass of the system is $m = m_1 + m_2 + m_3 + \cdots + m_n$. By definition, the center of mass of such system is point M, such that

$$m_1 \cdot \overrightarrow{MX_1} + m_2 \cdot \overrightarrow{MX_2} + m_3 \cdot \overrightarrow{MX_3} + \cdots + m_n \cdot \overrightarrow{MX_n} = 0$$

For the case of just two massive points, $\{m_1, X_1\}$ and $\{m_2, X_2\}$ this reduces to $m_1 \cdot \overrightarrow{MX_1} = -m_2 \cdot \overrightarrow{MX_2}$, the Archimedes famous lever rule.

Heuristic properties of the Center of Mass.



1. Every system of finite number of point masses has unique center of mass (COM).
2. For two point masses, m_1 and m_2, the COM belongs to the segment connecting these points; its position is determined by the Archimedes lever rule: the point's mass times the distance from it to the COM is the same for both points, $m_1 d_1 = m_2 d_2$.
3. The position of the system's center of mass does not change if we move any subset of point masses in the system to the center of mass of this subset. In other words, we can replace any number of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.

Given the coordinate system with the origin O, we can specify position of any geometric point A by the vector, $\overrightarrow{OA}$ connecting the origin O with this point. For the system of point masses, $m_1, m_2, m_3, \dots, m_n$, located at geometric points $X_1, X_2, X_3, \dots, X_n$, position of a point mass m_i is specified by the vector $\overrightarrow{OX_i}$ connecting the origin with point X_i where the mass is located.

It can be easily proven using the COM definition given above that the position of the COM of the system, M, is given by

$\overrightarrow{OM} = \frac{m_1 \cdot \overrightarrow{OX_1} + m_2 \cdot \overrightarrow{OX_2} + m_3 \cdot \overrightarrow{OX_3} + \cdots + m_n \cdot \overrightarrow{OX_n}}{m_1 + m_2 + m_3 + \cdots + m_n}$, or,

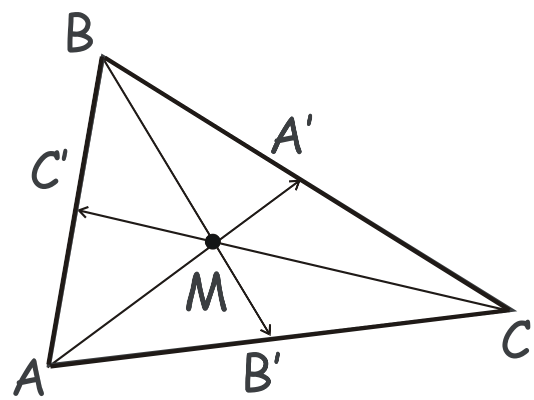
$$\overrightarrow{OM} = \frac{m_1 \cdot \overrightarrow{OX_1} + m_2 \cdot \overrightarrow{OX_2} + m_3 \cdot \overrightarrow{OX_3} + \cdots + m_n \cdot \overrightarrow{OX_n}}{m}$$

An important property of the COM immediately follows from the above. If we add a point (m_{n+1}, X_{n+1}) to the system, the resultant COM is the COM of the system of two points: the new point and the point $(m\,, M)$ with mass m placed at the COM of the first n points,

$$\overrightarrow{OM}^{(n+1)} = \frac{m \cdot \overrightarrow{OM} + m_{n+1} \cdot \overrightarrow{OX_{n+1}}}{m + m_{n+1}}$$

$$\overrightarrow{OM}^{(n+1)} = \frac{m_1 \cdot \overrightarrow{OX_1} + m_2 \cdot \overrightarrow{OX_2} + m_3 \cdot \overrightarrow{OX_3} + \cdots + m_n \cdot \overrightarrow{OX_n} + m_{n+1} \cdot \overrightarrow{OX_{n+1}}}{m_1 + m_2 + m_3 + \cdots + m_n + m_{n+1}}$$

Problem. Prove that the medians of an arbitrary triangle ABC are concurrent (cross at the same point M).



Problem. Prove that the altitudes of an arbitrary triangle ABC are concurrent (cross at the same point H).

Problem. Prove that the bisectors of an arbitrary triangle ABC are concurrent (cross at the same point O).

Problem. Prove Ceva's theorem.