

Homework for April 24, 2022.

Algebra/Geometry. Complex numbers.

Please, complete the previous homework assignments. Review the classwork handout on complex numbers. Solve the following problems.

Problems.

- (i) find the magnitude and the argument, (ii) compute the inverse, and (iii) find the magnitude and the argument of the inverse for the following complex numbers:
 - $1 + i$
 - $-i$
 - $1 + ix$
 - $\frac{\sqrt{3}}{2} + \frac{i}{2}$
 - $\frac{1}{2-i} - \frac{1}{2+i}$
- Find a complex number z whose magnitude is 2 and the argument $Arg(z) = \frac{\pi}{4} = 45^\circ$.
- Draw the following sets of points on complex plane.
 - $\{z | Re(z) = 1\}$
 - $\{z | Arg(z) = \frac{3\pi}{4} = 135^\circ\}$
 - $\{z | |z| = 1\}$
 - $\{z | Re(z^2) = 0\}$
 - $\{z | |z^2| = 2\}$
 - $\{z | |z - 1| = 1\}$
 - $\{z | z + \bar{z} = 1\}$
- Prove that for any complex number z , we have
 - $|\bar{z}| = |z|, Arg(\bar{z}) = -Arg(z)$
 - $\frac{\bar{z}}{z}$ has magnitude 1; check this for $z = 1 - i$.
- If z has magnitude 2 and argument $\frac{\pi}{2}$ and w has magnitude 3 and argument $\frac{\pi}{3}$, what will be the magnitude and the argument of zw ? Write it in the form $a + bi$.
- Let $P(x)$ be a polynomial with real coefficients.
 - Prove that for any complex number z , we have $\overline{P(z)} = P(\bar{z})$

- b. Let z be a complex root of this polynomial, $P(z) = 0$. Prove that then \bar{z} is also a root, $P(\bar{z}) = 0$.
7. Solve the equation $x^3 - 4x^2 + 6x - 4 = 0$. Find the sum and product of the roots in two ways: by using Vieta formulas and by explicit computation. Check that the results match.
8. Simplify following expression:
- $(1 + \sin \alpha)(1 - \sin \alpha)$
 - $(1 + \cos \alpha)(1 - \cos \alpha)$
 - $\sin^4 \alpha - \cos^4 \alpha$
9. Prove the following equalities:
- $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$
 - $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$
 - $\cos 4\alpha = 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1$
 - $\sin 4\alpha = 4 \sin \alpha \cos^3 \alpha - 4 \cos \alpha \sin^3 \alpha$
 - $\sin 5\alpha = 16 \sin^5 \alpha - 20 \sin^3 \alpha + 5 \sin \alpha$
 - $\cos 5\alpha = \dots$ (find the expression)
10. Solve the following equation:
- $\cos^2 \pi x + 4 \sin \pi x + 4 = 0$
11. Solve the following equations and inequalities:
- $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
 - $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$
 - $\sin^2 x - 2 \sin x \cos x = 3 \cos^2 x$
 - $\sin 6x + 2 = 2 \cos 4x$
 - $\cot x - \tan x = \sin x + \cos x$
 - $\sin x \geq \pi/2$
 - $\sin x \leq \cos x$

Algebra: polynomials recap.

Review the classwork notes on polynomials, factorization and Vieta theorem. Solve the following problems; skip those you already solved.

Problems.

- Write Vieta formulae for the cubic equation, $x^3 + Px^2 + Qx + R = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of P, Q and R ,
 - $(x_1 + x_2 + x_3)^2$
 - $x_1^2 + x_2^2 + x_3^2$
 - $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
 - $(x_1 + x_2 + x_3)^3$

- The three real numbers x, y, z , satisfy the equations

$$x + y + z = 6$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$$

$$xy + yz + zx = 11$$

- Find a cubic polynomial whose roots are x, y, z
 - Find x, y, z
- Find two numbers u, v such that

$$u + v = 6$$

$$uv = 13$$

- Find three numbers, a, b, c , such that

$$a + b + c = 2$$

$$ab + bc + ca = -7$$

$$abc = -14$$

- Find all real roots of the following polynomial and factor it.

- a. $x^8 + x^4 + 1$
 - b. $x^4 - x^3 + 5x^2 - x - 6$
 - c. $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$
6. Perform the long division, finding the quotient and the remainder, on the following polynomials.
- a. $(x^3 - 3x^2 + 4) \div (x^2 + 1)$
 - b. $(x^3 - 3x^2 + 4) \div (x^2 - 1)$