MATH 9: ASSIGNMENT 1

SEPT 26, 2021

General information

Welcome to a new semester of math at SchoolNova!
Below is some information about the new semester.
Teacher's name: Alexander Kirillov. The best way to reach me is by email: `kirillov@schoolnova.org`.
All homework assignments will be posted on SchoolNova website, https://schoolnova.org
We plan to cover the following topics (not necessarily in that order):

1. Geometry: Vectors and center of mass. Ceva theorem.
2. Logic ans set theory: quantifiers and proofs. Mathematical induction. Functions and bijections. Infinite sets and cardinality.
3. Algebra: continuation of number theory; Fermat's little theorem, Euler's function and public key cryptography. Polynomials and roots. Complex numbers and their geometric interpretation; de Moivre's formula.

We assume that you are familiar with most of the material of Math 8 at Schoolnova, in particular, with basic set theory and logic operations, with basics of Euclidean geometry such as congruence tests for triangles or properties of parallelograms, and with number theory: divisibility, congruences, Euclid's algorithm. We might review these topics as needed.

Some review problems

1. Find all numbers between 1–1000 which are
   - divisible by 15 and
   - give remainder 3 upon division by 7
2. (a) Compute $2^{2016} \mod 5$
   (b) Compute $2^{2016} \mod 11$
   (c) Compute $2^{2016} \mod 55$
3. (a) Use Euclid's algorithm to compute $\gcd(24, 19)$
   (b) Find the inverse of 19 mod 24, i.e. solve $19e \equiv 1 \mod 24$.
4. Describe how one can divide a given segment into 5 equal parts using ruler and compass.
5. Let $ABCD$ be a quadrilateral. Show that then the four midpoints of the sides of this quadrilateral form a parallelogram.

   [Hint: in a triangle, line connecting midpoints of two sides is parallel to the third side.]

Vectors

A **vector** is a directed segment. We denote the vector from A to B by $\overrightarrow{AB}$. We will also frequently use lower-case letters for vectors: $\vec{v}$.

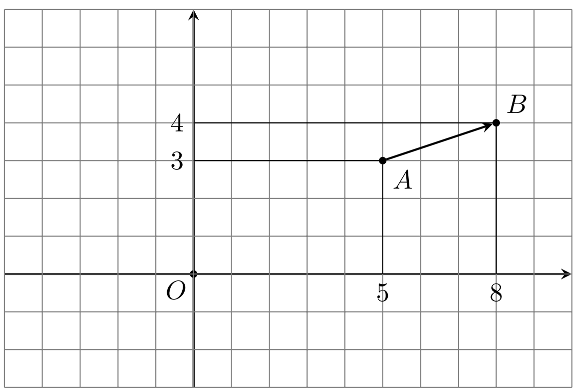
We will consider two vectors to be the same if they have the same length and direction; this happens exactly when these two vectors form two opposite sides of a parallelogram. Using this, we can write any vector $\vec{v}$ as a vector with tail at given point A. We will sometimes write $A+\vec{v}$ for the head of such a vector.

Vectors are used in many places. For example, many physical quantities (velocities, forces, etc) are naturally described by vectors.

Vectors in coordinates

Recall that every point in the plane can be described by a pair of numbers – its coordinates. Similarly, any vector can be described by two numbers, its x-coordinate and y-coordinate: for a vector $\overrightarrow{AB}$, with tail $A=(x_1,y_1)$ and head $B=(x_2,y_2)$, its coordinates are

$$\overrightarrow{AB}=(x_2-x_1,y_2-y_1)$$



$$\overrightarrow{AB}=(8-5,4-3)=(3,1)$$

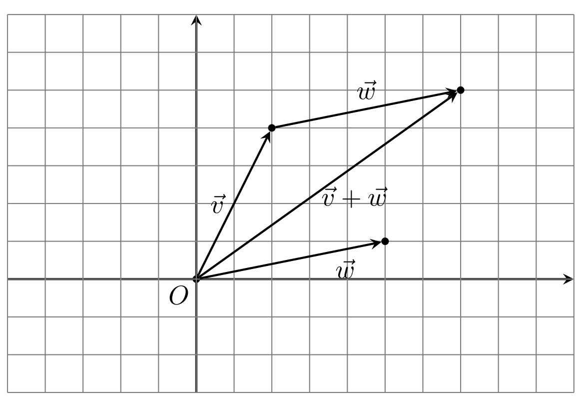
Operations with vectors

Let $\vec{v}$, $\vec{w}$ be two vectors. Then we define a new vector, $\vec{v}+\vec{w}$ as follows: choose A,B,C so that $\vec{v}=\overrightarrow{AB}$, $\vec{w}=\overrightarrow{BC}$; then define

$$\vec{v}+\vec{w}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

In coordinates, it looks very simple: if $\vec{v}=(v_x,v_y)$, $\vec{w}=(w_x,w_y)$, then

$$\vec{v}+\vec{w}=(v_x+w_x,v_y+w_y)$$



Theorem. *So defined addition is commutative and associative:*

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

$$(\vec{v_1} + \vec{v_2}) + \vec{v_3} = \vec{v_1} + (\vec{v_2} + \vec{v_3})$$

There is no obvious way of multiplying two vectors; however, one can multiply a vector by a number: if $\vec{v} = (v_x, v_y)$ and t is a real number, then we define

$$t\vec{v} = (tv_x, tv_y)$$

Again, we have the usual distributive properties: $t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$.

Problems

1. Let $A = (x_1, y_1)$,$B = (x_2, y_2)$. Show that the midpoint M of segment AB has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ and that $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.

[Hint: point M is $A + \frac{1}{2}\vec{v}$, where $\vec{v} = \overrightarrow{AB}$].

2. Let AB be a segment, and M -– a point on the segment which divides it in the proportion 2:1, i.e., $|AM| = 2|MB|$. Let O be the origin. Show that $\overrightarrow{OM} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OB}$

3. Consider a parallelogram $ABCD$ and let $\vec{v} = \overrightarrow{AB}$, $\vec{w} = \overrightarrow{AD}$. Write the following vectors as combinations of $\vec{v}, \vec{w}$:

(a) $\overrightarrow{AC}$

(b) $\overrightarrow{BD}$

(c) $\overrightarrow{AM}$, where M is the midpoint of BD

Using this, can you prove that midpoint of BD coincides with the midpoint of AC?

4. Consider triangle ABC and let AA_1, BB_1, CC_1 be the medians of the triangle.

(a) Write $\overrightarrow{AA_1}$ as combination of vectors $\vec{v} = \overrightarrow{OA}$, $\vec{w} = \overrightarrow{OB}$, $\vec{u} = \overrightarrow{OC}$.

(b) Prove that then $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = 0$.

5. (a) In the notation of the previous problem, let M_1 be the point that divides median AA_1 in proportion $2 : 1$ (i.e., $AM_1 : M_1A_1 = 2 : 1$). Write vector $\overrightarrow{OM_1}$ as a combination of $\vec{v}, \vec{w}, \vec{u}$. Repeat the same for all other medians.

(b) Prove that the three medians of a triangle intersect at a single point.