MATH 9: REVIEW OF LOGIC

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1. Logic Review

In logic, we work with **statements**: something that can be either true or false. In Math 8, we had discussed various logic operations:

- Logical AND: $P \wedge Q$ — true if both P, Q are true
- Logical OR: $P \vee Q$ — true if at least one of P, Q is true. Note that this includes the case when both P,Q are true.
- Negation: $\neg P$ — true if P is false, and false if P is true.
- Implication: $P \implies Q$. By definition, it is true in all cases except one: if P is true and Q is false.

These operations satisfy a number of logic laws, some of them listed below. All of them can be obtained from truth tables. The most important laws are

- **Modus Ponens**: given P and $P \implies Q$, we can conclude Q. In formulas, it can be written as
$$(P \wedge (P \implies Q)) \implies Q$$
- Transitivity of implication:
$$(P \implies Q \wedge Q \implies R) \implies (P \implies R)$$
- **Modus Nolens**: if $P \implies Q$, and we know that Q is false, then P is also false
$$((P \implies Q) \wedge \neg Q) \implies \neg P$$

2. Quantifiers

We write the statement $\forall x\ P(x)$ to mean *for all values of x, $P(x)$ is true*, and $\exists x\ P(x)$ to mean *there is some value of x such that $P(x)$ is true*. Here $P(x)$ is statement which depends on some variable x.

In all of this, we assume that we know what set of values x can take (e.g., x is a real number). Sometimes we include the set of values in the quantifier: $\forall x \in \mathbb{Z}\ x^2 > 0$.

The following property is used very frequently:

$$\neg(\forall x\ P(x)) = \exists x\ \neg(P(x))$$
$$\neg(\exists x\ P(x)) = \forall x\ \neg(P(x))$$

3. Proofs

Conditional proof: To prove $A \implies B$, assume that A is true and then deduce B.

Proof by contradiction: To prove some logical statement or theorem T, assume that T is false (or assume that the negation of T is true) and deduce a logical contradiction (e.g. a statement of the form $A \wedge \neg A$).

Combination of the above: To prove $A \implies B$, assume that A is true and that B is false and then derive a contradiction.

Proof of existence: To prove a statement of the form $\exists x P(x)$, it is enough to find one example of x for which $P(x)$ holds true.

Proofs of universal quantifier: To prove a statement of the form $\forall x P(x)$, you must provide a general argument which works for all possible values of x, that deduces that $P(x)$ is true.

To prove any statement with a quantifier, it may sometimes be easier to use a proof by contradiction. For example, given $0 \neq 1$, can you prove $\forall x(x \neq 0 \vee x \neq 1)$? Try writing out the logical negation of the statement and go from there.

Homework

1. You are given the following statements:

$A \wedge B \implies C$
$B \vee D$
$C \vee \neg D$

Using this, deduce $A \implies C$.

2. Let $\mathbb{R}$ be the set of all integers. Prove $\forall x \in \mathbb{R} : x^2 + 2x + 4 > 0$.

You can use without proof some basic properties of arithmetic operations, such as associativity, commutativity, and distributivity; you can also use the fact that product of positive numbers is positive.

3. Let $\mathbb{Z}$ be the set of all integers. Are the following statements true or false? Can you prove them?

(a) $\forall n \in \mathbb{Z}\ n^2 > 0$
(b) $\exists n \in \mathbb{Z}\ n^2 > 0$
(c) $\forall n \in \mathbb{Z}(n^3 > 0 \implies n > 0)$

4. Recall that an integer number n is called even if $\exists k \in \mathbb{Z}\ n = 2k$. We will use notation $E(n)$ for statement n is even.

Prove the following:

(a) If n, m are even, then $m + n$ is even.
(b) If m is even and $m + n$ is even, then n is even
(c) If n is even, then for any m, mn is even.
(d) For any n, the number $n(n + 1)$ is even.

5. Write each of the following statements using only quantifiers, arithmetic operations, equalities and inequalities. In all problems, letters x, y, z stand for a variables that takes real values, and letters $m, n, k, \ldots$ stand for variables that take integer values.

(a) Equation $x^2 + x - 1$ has a solution
(b) Inequality $y^3 + 3y + 1 < 0$ has a solution
(c) Inequality $y^3 + 3y + 1 < 0$ has a positive real solution
(d) Number 100 is even.
(e) Number 100 is odd
(f) For any integer number, if it is even, then its square is also even.

6. For each of the statements of the previous problem, try to determine if it is true. If it is, give a proof. If not, disprove it (i.e., prove its negation).

***7.** Consider the following arguments:

(a) No homework is fun.
Some reading is homework.
Therefore, some reading is not fun.

(b) All informative things are useful.
Some websites are not useful.
Some websites are not informative.

For each of them,

(1) write it in symbolic form, using quantifiers. Use set A for set of all human activities (which includes reading, writing, homework, etc) and notation $F(x)$ for "x is fun", $H(x)$ for "x is homework", etc) and

(2) prove it, using the methods of proof discussed in class.

FYI: these are examples of two of Aristotle's syllogisms, namely *Ferio* and *Baroco*. For more information, google Aristotle and syllogism.