MATH 9
ASSIGNMENT 6: MATHEMATICAL INDUCTION CONTINUED

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Mathematical induction

Recall from last time the following result, called the **Principle of Mathematical Induction**:

Let $P(n)$ be a statement which depends on a natural number n (natural numbers are nonnegative integers). Suppose that we know the following:

- $P(0)$ is true
- For every n, the statement $\Big(P(n) \implies P(n+1)\Big)$ is true.

Then $P(n)$ is true for all n.

Proving that $P(0)$ is true is called the **base case**.

Proving the implication $P(n) \implies P(n+1)$ is called the **inductive step**. It is important to understand that it is the implication itself you are proving, not either of the statements $P(n)$ or $P(n+1)$. In other words, you are proving that **if** $P(n)$ is true, **then** $P(n+1)$ is also true.

A variation of the principle of mathematical induction is when instead of taking the base case to be $n = 0$, you take the base case $n = 1$ (or some other number n_0); in this case, mathematical induction establishes that the statement is true for all $n \geq n_0$.

Full induction

Sometimes it is more convenient to use the following version of induction principle, called **Full induction**. It is easily shown to be equivalent to the original one.

Let $P(n)$ be a statement which depends on a natural number n (natural numbers are nonnegative integers). Suppose that we know the following:

- $P(0)$ is true
- For every $n \geq 0$, if all of the statements $P(0)$, $P(1)$, …, $P(n)$ are true, then $P(n+1)$ is also true.

Then $P(n)$ is true for all n.

Well ordering

Yet one more version is the following, called **well ordering principle**. Recall that natural numbers are non-negative integers.

Theorem. *In any non-empty set of natural numbers there is a smallest element.*

The usual way this theorem is used is as follows. Suppose we want to prove that all natural numbers have some property P. Assume that it is not so, i.e. there are natural numbers that do not have this property. Let k be the smallest of them; then by assumption, all natural numbers smaller than k have this property. Now get a contradiction.

Homework

1. Let Fibonacci numbers F_n, $n \geq 1$, be defined by the following rules:
- $F_1 = F_2 = 1$
- For all $n \geq 2$, $F_{n+1} = F_n + F_{n-1}$

(a) Write the first 10 Fibonacci numbers

(b) Let $S_n = F_1 + F_2 + \cdots + F_n$. Compute S_n for several first values of n. Guess the formula for S_n and prove it using induction.

2. A real number x is such that $x + \frac{1}{x}$ is integer. Prove that then, for any $n \geq 1$, $x^n + \frac{1}{x^n}$ is also integer.

3. Show that for any $n \geq 1$, the following inequality holds:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

4. A sequence x_n is defined by rules $x_1 = 5$, $x_{n+1} = 2x_n - 3$. Write down first eight terms; try to guess the formula for x_n and prove it using induction. [Hint: compare x_n with powers of 2.]

5. We normally take it for granted that we have division with remainder for integers: given natural numbers $n \geq 0, d > 1$, one can always find q, r such that

$$n = qd + r, \qquad 0 \leq r < d$$

Can you give a rigorous proof of this fact by induction in n? You can use any form of induction (e.g. full induction, well-ordering principle).

Hint: look at number $n - d$.

6. Show that if we draw n lines on a plane, then we can color each of the regions formed by these lines black or white so that regions that have a common boundary have different colors. [This problem has more than one solution]

***7.** If we draw n lines on the plane so that no two of them are parallel, and no three go through the same point, into how many regions do they divide the plane?