MATH 9
ASSIGNMENT 7: POLYNOMIALS AND ROOTS

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Polynomials: basics

A polynomial (in variable x) is an expression of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where a_i are real numbers. (Later we will also consider other possible coefficients.)

The highest power of x appearing in $p(x)$ is called **degree** of $p(x)$ and is denoted $\deg p(x)$; the coefficient of the highest power of x is called the **leading coefficient**. In particular, every number can be considered as a polynomial of degree zero.

The set of all polynomials in variable x is denoted $\mathbb{R}[x]$.

Polynomials can be added, subtracted, and multiplied. It is immediate from definition that if $p(x)$, $q(x)$ are polynomials of degree $\le n$, then $p \pm q$ is also a polynomial of degree $\le n$. Moreover,

$$\deg p(x)q(x) = \deg p(x) + \deg(q(x)).$$

Polynomial division

As with integers, in general, we can not divide one polynomial by another and expect to get a polynomial.

We say that $f(x)$ is **divisible** by $g(x)$ if there exists a polynomial $q(x)$ such that $f(x) = g(x)q(x)$. Note that it in this case, $\deg f(x) \ge \deg g(x)$.

If the polynomials are not divisible we have **division with remainder**, also known **long division**.

Theorem. *Given polynomials $f(x)$, $g(x)$ (with degree of $g(x)$ at least 1), one can uniquely write $f(x)$ in the form*

$$f(x) = q(x)g(x) + r(x), \quad \deg r(x) < \deg g(x)$$

Polynomials $q(x)$, $r(x)$ are called quotient *and* remainder *respectively.*

Moreover, if $f(x)$, $g(x)$ have integer coefficients, and the leading coefficient of $g(x)$ is equal to 1, then $q(x)$, $r(x)$ also have integer coefficients.

Proof. Proof goes by induction in $n = \deg f(x)$. Details were given in class. □

Explicit algorithm for this division has been introduced in class.

Roots and Bezout theorem

A number $c \in \mathbb{R}$ is called a **root** of polynomial $p(x)$ if $p(c) = 0$.

Theorem (Bezout theorem)**.** *When a polynomial $p(x)$ is divided by $(x - c)$, the remainder is $p(c)$. In particular, $p(x)$ is divisible by $(x - c)$ if and only if c is a root.*

Proof. Using long division, write $p(x) = (x - c)q(x) + r(x)$. Since $(x - c)$ has degree 1, the remainder $r(x)$ must have degree zero, i.e. be a number: $p(x) = (x - c)q(x) + r$. Now substituting in this equation $x = c$, we get $p(c) = r$. □

More generally, it can be shown that if $c_1, \dots, c_n$ are distinct roots of $p(x)$, then $p(x)$ is divisible by the product $(x - c_1)\dots(x - c_n)$ (see homework problem 3)

Problems

1. Use the long division to find the quotient and remainder for the following division problems:
(a) $(x^3 - 12x^2 - 42) \div (x^2 - 1)$
(b) $(x^{13} + 1) \div (x - 2)$
*(c) $x^{81} + x^{49} + x^{25} + x^9 + x \div x^3 - x$

2. (a) Show that for any n, $x^n - 1$ is divisible by $x - 1$. Find the quotient.
(b) Show that $x^n + 1$ is divisible by $x + 1$ if and only if n is odd. Find the quotient.

3. Prove that if $c_1, \dots, c_n$ are distinct roots of $p(x)$, then $p(x)$ is divisible by the product $(x-c_1)\dots(x-c_n)$. [Hint: if c_n is a root, then $p(x) = (x - c_n)q(x)$. Now use induction.]

Deduce from this that if a polynomial of degree $\leq n$ has value 0 at $n+1$ different points, then this polynomial must be identically zero (i.e. have all coefficients zero).

4. The polynomial $P(x)$ has remiander 99 when divided by $x - 19$ and remainder 19 when divided by $x - 99$. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

5. Let $P(x)$ be a polynomial with integer coefficients and let a, b be integers, $a \neq b$. Prove that then $P(a) - P(b)$ is divisible by $(a - b)$.

6. Is it possible to find a polynomial with integer coefficients such that $P(7) = 11$ and $P(11) = 13$?

7. Prove that $x^{2n} + x^n + 1$ is divisible by $x^2 + x + 1$ if and only if n is not a multiple of 3.

8. Is it true that if the polynomial $P(x)$ is such that $P(n)$ is an integer for any integer n, then $P(x)$ has integer coefficients?

9. Construct a quadratic polynomial $f(x)$ such that $f(-1) = 1$, $f(0) = 0$, $f(2) = 4$.

***10.** Does there exist a polynomial with integer coefficients $P(x)$ such that for every integer n, $P(n)$ is a prime number?