MATH 9
ASSIGNMENT 9: INTERPOLATION POLYNOMIALS

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Reminder

Definition. A number a is called a multiple root of a polynomial $f(x)$, with multiplicity m, if $f(x)$ is divisible by $(x-a)^m$ and not divisible by $(x-a)^{m+1}$.

Roots of multiplicity one are also called simple roots; of multiplicity two, double roots.

For example, polynomial $(x-1)^2(x-5)$ has a simple root $x=5$ and double root $x=1$.

Theorem. *If $x_1, \dots, x_n$ are roots of polynomial $f(x)$, listed with multiplicities, then $f(x)$ is divisible by $(x-x_1)\dots(x-x_n)$.*

This also implies the following result.

Theorem. *A non-zero polynomial of degree n can not have more than n roots (counting with multiplicities, i.e. counting each root as many times as is its multiplicity).*

Interpolation polynomials

Suppose you are given a function $f(x)$ about which you do not know much, but you do know its value at several points: $f(x_1)=y_1, \dots, f(x_n)=y_n$. It is frequently needed to "approximate" f, i.e. find a nice function (say, a polynomial) which has the same values at these points. Here is one way to do it.

Theorem. *Given n distinct points on the real line $x_1, \dots, x_n$ and n (arbitrary) numbers $y_1, \dots, y_n$, there is exactly one polynomial of degree $n-1$ such that*

$$f(x_1)=y_1, \dots, f(x_n)=y_n$$

Proof. **Uniqueness.** If f_1, f_2 are two such polynomials, then f_1-f_2 will have zeros at $x_1, \dots, x_n$. Since f_1-f_2 has degree at most $n-1$, this implies $f_1-f_2=0$.

Existence. Idea: let us write f as a sum $y_1f_1+\dots+y_nf_n$, where each f_i is equal to 1 at x_i and zero at all other x_j. Clearly, so constructed f will satisfy the requirement:

$$f(x_i)=y_1f_1(x_i)+\dots+y_nf_n(x_i)=y_if_i(x_i)=y_i$$

How does one construct polynomials f_i? For example, how do we construct f_1?

Note that $g_1=(x-x_2)\dots(x-x_n)$ has the property $g(x_j)=0$ for all $j=2,\dots,n$ — so it meets parts of the requirement. But $g(x_1)=(x_1-x_2)\dots(x_1-x_n)\neq 1$. So let us take

$$f_1(x)=\frac{g_1(x)}{g_1(x_1)}=\frac{(x-x_2)\dots(x-x_n)}{(x_1-x_2)\dots(x_1-x_n)}$$

(note that the denominator is a number, not a polynomial). Then

$$f_1(x_1)=1, \quad f_1(x_j)=0 \quad \text{for } j\neq 1$$

In a similar way we construct all other f_i (exercise!) so that

$$f_i(x_i)=1, \quad f_i(x_j)=0 \quad \text{for } j\neq i$$

and then write $f(x)=y_1f_1(x)+\dots+y_nf_n(x)$. □

Problems

1. (a) Find a quadratic polynomial $f(x)$ such that $f(-1) = 7$, $f(1) = 5$, $f(2) = 10$.

(b) We know that a certain function has these values: $f(-1) = 2$, $f(0) = 1$, $f(2) = 5$, $f(3) = 22$. Try approximating it by a polynomial of degree 3. What would you expect as the value of f at $x = 1$?

2. Simplify

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-a)(x-c)}{(b-a)(b-c)}$$

Can you do it without long computations?

3. A ship is sailing in a straight line with constant speed. Every hour, the captain measures the distance to a lighthouse on a small island which lies some distance off their path.

At noon, the distance was 7 km; at 2 pm, it was 5 km; at 3pm, it was 11 km.

What was the distance at 1 pm? What is the ship's speed?

[Hint: it might be easier to work with the square of the distance than with the distance itself.]

4. A polynomial $f(x)$, when divided by $(x-a)$, $(x-b)$, and $(x-c)$ gives remainders A, B, and C respectively. What is the remainder when $f(x)$ is divided by $(x-a)(x-b)(x-c)$?

[Hint: the remainder is a quadratic polynomial, so we can determine it uniquely if we know its values at 3 points.]

5. It is known that one of the roots of the polynomial $x^3 - 6x^2 + ax - 6$ is $x = 3$. Can you find all other roots?

6. Numbers x_1, x_2, x_3 satisfy the following system of equations:

$$x_1 + x_2 + x_3 = 7$$
$$\tfrac{1}{x_1} + \tfrac{1}{x_2} + \tfrac{1}{x_3} = \tfrac{1}{7}$$

Prove that one of these 3 numbers must be equal to 7.

[Hint: use Vieta formulas.]