# MATH 9
## ASSIGNMENT 19: COUNTABLE AND UNCOUNTABLE SETS

MAR 6, 2022

### Countable sets

Recall that an infinite set $A$ is called *countable* if there is a bijection between $A$ and the set $\mathbb{N}$ of positive integers. In other words, it means that elements of $A$ can be indexed by positive integers, so we can write

$$A = \{a_1, a_2, \dots\}$$

Then we have the following result:

- Set $\mathbb{Z}$ is countable
- Set $\mathbb{N} \times \mathbb{N}$ of pairs of positive integers is countable
- If $A$ is finite and $B$ is countable, then $A \cup B$ is countable

More examples of countable sets are given in problems 1–3 below.

### Continuum

There is another class of infinite sets: those which have "as many elements as" the set $\mathbb{R}$ of all real numbers, i.e. sets $A$ for which there exists a bijection $A \to \mathbb{R}$. Such sets are said to have continuum cardinality. Examples of these sets include (see problem 4):

- Set of all infinite sequences of zeros and ones
- Interval $[0, 1]$
- Set $\mathbb{R}$ of all real numbers

The question whether continuum cardinality is the same as countable (i.e., whether there is a bijection between $\mathbb{N}$ and $\mathbb{R}$) was left open today.

### Homework

**1.** Let $A, B$ be countable sets.
  (a) Prove that then $A \cup B$ is also countable.
  (b) Prove that then the set of pairs $A \times B = \{(a, b) \mid a \in A, b \in B\}$ is countable

**2.** Let $A_1, A_2, \dots$ be a collection of sets, each of them countable. Prove that the union $A_1 \cup A_2 \cup A_3 \cup \dots$ is also countable. [Hint: construct a bijection between this set and $\mathbb{N} \times \mathbb{N}$.]

**3.** Prove that each of the following sets is countable
  (a) Set $\mathbb{Q}$ of rational numbers
  (b) Set $\mathbb{Q} \times \mathbb{Q}$ of pairs of rational numbers
  (c) Set of all quadratic polynomials $ax^2 + bx + c$ with rational coefficients
  (d) Set of all polynomials with rational coefficients.

**4.** Show that each of the following sets has the same cardinality as $[0, 1]$ (i.e., there is a bijection between each of these sets and $[0, 1]$):
  (a) Interval $[0, 1)$ [Hint: interval $[0, 1]$ can be written as $[0, 1] = A \cup B$, where $A = \{1/n, n \in \mathbb{N}\}$ and $B$ is all other numbers in $[0, 1]$].
  (b) Open interval $(0, 1)$
  (c) Set of all infinite sequences of 0s and 1s
  (d) $\mathbb{R}$
  *(e) $[0, 1] \times [0, 1]$