Homework 5

Refraction at a spherical surface. Lensmaker's equation.

Let us consider refraction at a spherical surface. We have two media with refractive indices $\mathbf{n_1}$ and $\mathbf{n_2}$. The media are separated with a spherical boundary (Figure 1)

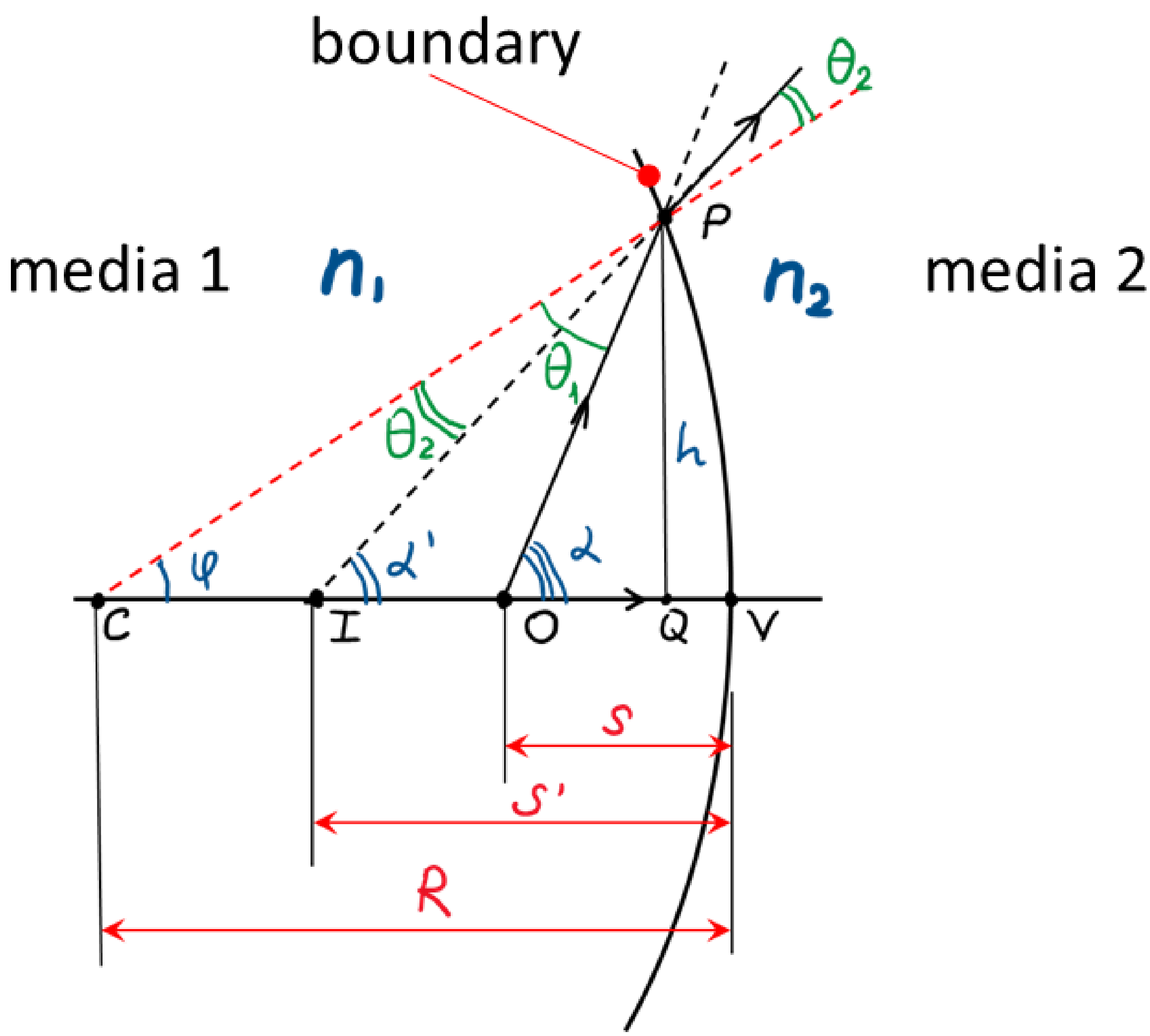

Figure 1. Refraction at a spherical surface.

We are going to find the image of point O. For this we will use two rays emanating from point O: OQ and OP. Ray OP is refracted at point P, the refraction angle is Θ_1. Point C is the center of the spherical surface, so CP is the radius of the sphere. As we know, CP is perpendicular to the sphere's tangent plane in point P. So, for ray OP, $\boldsymbol{\theta_1}$ is the angle of incidence and $\boldsymbol{\theta_2}$ is the angle of refraction. According to Snell's law we have:

$$n_1 sin\theta_1 = n_2 sin\theta_2 \qquad (1)$$

The refracted ray and the ray OQ appear to emerge from their common intersection, point **I** which is the image of point **O**. In triangle CPO, **α** is the exterior angle, so we have:

$$\alpha = \theta_1 + \varphi \qquad (2)$$

For triangle CPI we have:

$$\alpha' = \theta_2 + \varphi \qquad (3)$$

From (1), (2) and (3), we obtain:

$$n_1(\alpha - \varphi) = n_2(\alpha' - \varphi) \qquad (4)$$

Then we will neglect the distance QV (we assume that the radius R is much larger than h) and express the angles **α, α'** and **φ** through **s, s'** and **h** as:

$$\alpha \approx \frac{h}{s};\ \alpha' \approx \frac{h}{s'};\ \varphi \approx \frac{h}{R} \qquad (5)$$

Here we used the approximation $tg(\alpha) \approx \alpha$, which is valid for small angles. So we have:

$$n_1\left(\frac{h}{s} - \frac{h}{R}\right) = n_2\left(\frac{h}{s'} - \frac{h}{R}\right) \qquad (6)$$

or

$$\frac{n_1}{s} - \frac{n_2}{s'} = \frac{n_1 - n_2}{R} \qquad (7)$$

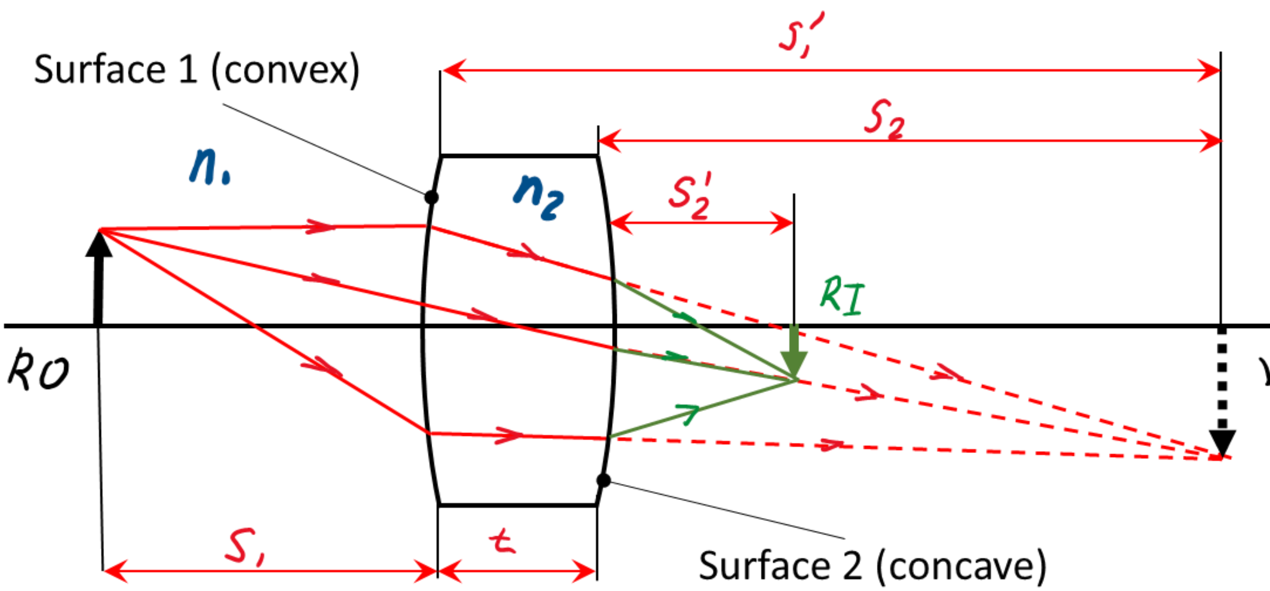

Figure 2. Ray diagram of a convex lens.

We can rewrite the expression (7) using the same sign convention we used for the spherical mirrors:

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \qquad (8)$$

This sign convention is: "***positive distances for real images and objects, negative distances for the virtual ones. Positive radius of curvature for convex surfaces and negative for concave ones.***"

Now let us consider area with a refraction index $\mathbf{n_2}$, bounded by 2 spherical surfaces with different radii of curvature $\boldsymbol{R_1}$, left and $\boldsymbol{R_2}$, right (Figure 2). This area works as a convex lens. The light rays emanated from object RO (real object) are refracted at the surface 1 (convex, as it is "seen" by the rays). After the refraction they converge as if they would meet at the tip of object VO, and we can write:

$$\frac{n_1}{s_1} + \frac{n_2}{s_1'} = \frac{n_2 - n_1}{R_1} \qquad (8)$$

But this does not happen because the rays are intercepted by the second surface which they "see" as a concave one. This "interception", of course, does not affect the refraction at the first surface. Note, that ion this time they come from the media with refractive index $\mathbf{n_2}$ to the media with a refractive index $\mathbf{n_1}$. After the refraction at the second surface the rays are focused and form a real image RI. What is the relation between the distance to the "would be" image VO and the real image RI? To answer this question let us consider just the second surface. We learned that if we would be able to reverse the time, the trajectories of the light rays would have stayed the same, but the propagation directions would have changed to the opposite ones. In other words, geometrical optics is "time reversible". So, let us reverse the time. In this case, the light rays will be emitted from the object RI and, after refraction at the second surface, they will diverge as if they would have come from VO, which, in this case, would work as a virtual image. So, we can write same expression as (8).

$$\frac{n_2}{s_2} + \frac{n_1}{s_2'} = \frac{n_1 - n_2}{R_2} \qquad (9)$$

We should keep in mind that in reality the rays go *toward* VO rather than from it, so VO works as a "***virtual object***". Now, the distance $\mathbf{s_2}$ to the virtual object is:

$$s_2 = -(s_1' - t) = t - s_1' \qquad (10)$$

We used negative sign since VO is a virtual object. In a "thin lens approximation", we can neglect $\boldsymbol{t}$ (later we will give more accurate criteria for the "thin lens"). In this case

$$s_2 \approx -s_1' \qquad (11)$$

Now we can substitute $\boldsymbol{-S_1}$ instead $\boldsymbol{S_2}$ in to equation (9) and add equations (8) and (9).:

$$\frac{n_1}{s_1}+\frac{n_1}{s_2'}=(n_2-n_1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad (12)$$

Now S_1 is the original object distance S_o and S_2' is the final image distance S_i so, after dividing both parts of the equation (12) by n_1 we have:

$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{n_2-n_1}{n_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad (13)$$

The *focal length* ***f*** of the thin lens is defined as the image distance for an object at infinity (in this case the light rays from the object are parallel). If $s_o \to \infty$, then $1/s_i \to 0$, and $s_i \to f$, so we have:

$$\frac{1}{f}=\frac{n_2-n_1}{n_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad (14)$$

Equation 14 is called "***lensmaker's equation***". This equation makes it possible to calculate focal distance of the lens if you know the radii of curvature of the sides and refractive indices of the lens material (n_2) and the media where the lens is used (n_1) (Figure3). Equation (13) can be then rewritten as:

$$\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f} \qquad (15)$$

This is the same equation we used for the reflection at spherical surfaces. Now we can give a more accurate definition of the thin lens: a lens is considered "thin" if the thickness of the lens is much less than its focal distance.

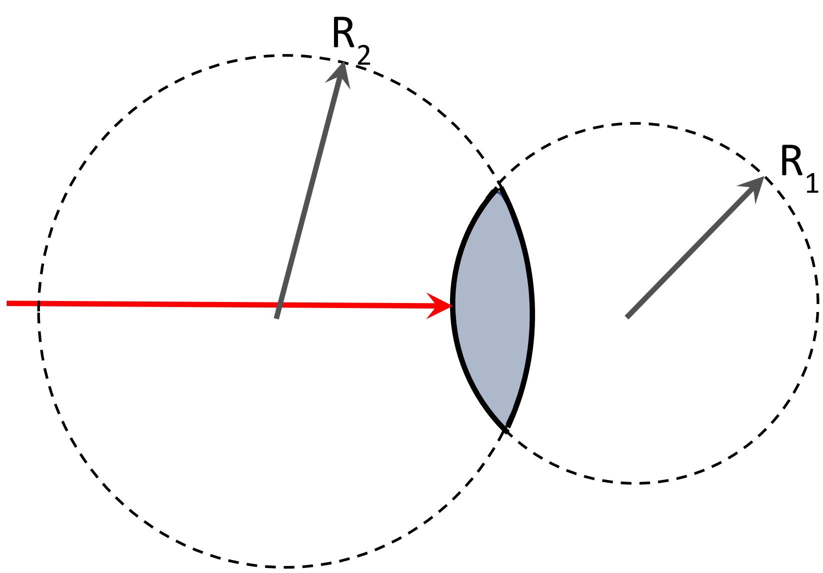

Figure 3. Biconvex lens

For the biconvex lens shown in Figure 1 light moving from left to right first "meets' the convex surface with radius of curvature R_1. Then, moving inside the lens, the light "meets" another surface, concave, with radius of curvature R_2. So in the lensmaker's equation for a biconvex lens R_2 has to be taken with "minus" sign.

Problems:

1. Does the focal distance of a lens change if we put the lens it in water?

2. Can same lens work as a converging or diverging lens depending on the substanceit is placed in?

3. Find a formula for the focal distance of a symmetrical biconvex lens with both radii of curvature equal to R and the lens material refractive index in n. The lens will be used in air (n air $\approx$ 1). Compare this formula with the formula for the focal distance of a spherical mirror with the same radius of curvature.

4. In a thick glass plate there is a hollow shaped as a convex lens. Will this "lens" work as converging or diverging? Prove your answer.

5. A convex-concave lens (see fig. below) has radii of curvature R=10cm and 3R. Find refractive index of the lens material if its focal length is 20 cm. (The lens is in air, n=1)