Homework 17

Dependence of energy on momentum in special relativity.

During last class we discussed the connection between energy and momentum in special relativity. The expression for relativistic momentum is similar to the classical one, except we use relativistic mass instead of the rest mass:

$$\vec{p} = m\vec{v} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \vec{v} \qquad (1)$$

As the speed tends to zero, $\boldsymbol{v^2/c^2}$ is $\boldsymbol{<<1}$ and we can neglect this term in the expression (1), which becomes reduced to familiar classical formula for the momentum. The relativistic energy is:

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (2)$$

Again, as the speed is zero, the energy becomes:

$$E = m_0 c^2 \qquad (3)$$

This part of energy is called “rest energy”. The rest energy is, by definition, the energy of a particle at rest, when the velocity of the particle equals 0. Kinetic energy $\boldsymbol{K}$ of the particle is the part of energy which depends on the particle’s speed, so it is the difference between total energy determined by expression (2) and the rest energy:

$$K = m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \right) \qquad (4)$$

In case the speed is very very small, but nonzero, the expression (4) can be simplified using the following mathematical approximation:

$$If \;\; x\ close\ to\ 0, \;\; then \quad \frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{x^2}{2} \qquad (5)$$

You can check it with a calculator. This approximation is based on one of the basic principles of mathematical analysis: a complicated function can be approximated by a simpler one (in our case it is a parabola) if argument of the complicated function is close to a certain value (in our case it is x=0). In other words, in the vicinity of the point x=0, the graph of our function $y = \frac{1}{\sqrt{1-x^2}}$ is almost identical to the graph of the parabola $= 1 + \frac{x^2}{2}$. There is a special mathematical procedure to choose this simpler function, but I will not give it here.

Using (4) and (5)we will obtain:

$$K = m_0c^2\left(1 + \frac{v^2}{2c^2} - 1\right) = \frac{m_0v^2}{2} \qquad (6)$$

This expression is the familiar classic formula for kinetic energy.

How does the relativistic energy related to the relativistic momentum? In classical mechanics this relation is simple:

$$K = \frac{m_0v^2}{2} = \frac{m_0^2v^2}{2m_0} = \frac{p^2}{2m} \qquad (7)$$

In special relativity it is more complicated. From (1) and (2) we can see that

$$E^2 = (K + m_0c^2)^2 = (pc)^2 + (m_0c^2)^2 \qquad (8)$$

Opening the parenthesis in the left side we obtain:

$$K^2 + 2Km_0c^2 = p^2c^2 \qquad (9)$$

If $v \ll c$, then the kinetic energy of a moving particle is much less than its rest energy m_0c^2, so $K^2 \ll 2Km_0c^2$ and we can neglect K^2 in formula (9). This leads us to the classical formula (7). The relation between energy, kinetic energy and momentum in special relativity is represented using the following mnemonic device given in “Basic concepts in relativity and early quantum theory” by R.Resnick and D.Halliday: (Figure 1). The expressions connecting the relativistic energy, kinetic energy and momentum can be obtained from the right triangle shown in Figure 1 using Pythagorean relation.

Also, $\sin\theta = \frac{v^2}{c^2}$ and $\sin\phi = \sqrt{1 - \frac{v^2}{c^2}}$.

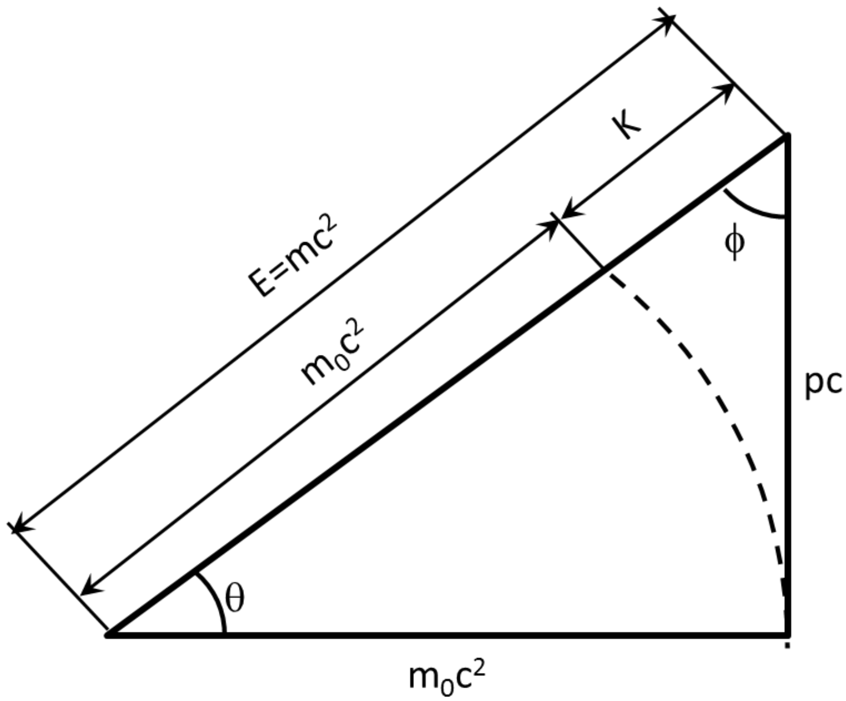

Figure 1.

Problems:

1. Consider a building brick, of volume V_0 and rest mass m_0, at rest in a certain reference frame. The brick is viewed by a second observer for whom it is moving with speed u along the brick long side. What will be the brick density ρ from the point of view of this observer in terms of the brick rest density ρ_o?

2. How much work can be done to increase the speed of an electron from rest to 0.5c?