Homework18

Relativistic Doppler effect

Let us consider a distant star which is moving toward us at a constant velocity $\boldsymbol{V}$. The star emits light which we can see. As long as we remember, the light is a wave and the "waving" parameters are electric and magnetic field strengths. The frequency of light determines its color (for the visible light), the frequency of the blue light is higher than the frequency of the red one. In the reference frame of the star (the reference frame which is moving with the star, so the star there is at rest) the electric field of the light can be written as:

$$E(x', t') = E_0 \cos(k'x' - \omega' t') \qquad (1)$$

Here "prime" indicates that the parameter is taken in the moving reference frame, $k' = 2\pi/\lambda'$ is the wave number, $\lambda'$is the wavelength, $\omega' = 2\pi c/\lambda'$ is the angular frequency and c is the speed of light. We are going to see how does the frequency of light changes in our reference frame. To find this out, we will use the Lorentz transformations which connects the coordinates and time in both the reference frames:

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}; \qquad t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad (2)$$

If we substitute *x'* and *t'* in the equation (1), and regroup the terms we will have:

$$E(x', t') = E_0 \cos\left(\underbrace{\left[\frac{k' + \omega' V/c^2}{\sqrt{1 - V^2/c^2}}\right]}_{K} x - \underbrace{\left[\frac{\omega' + k'V}{\sqrt{1 - V^2/c^2}}\right]}_{\omega} t\right) \qquad (3)$$

The formulas in the rectangular parenthesis are wave number k and angular frequency in our (rest) reference frame. Let us take a close look at the equation for the angular frequency:

$$\omega = \left[\frac{\omega' + k'V}{\sqrt{1 - V^2/c^2}}\right] \qquad (4)$$

Let us substitute k' as ω'/c, since the speed of light is the same in all the reference frames. Then we obtain:

$$\omega = \left[\frac{\omega' + \omega' V/c}{\sqrt{1 - V^2/c^2}}\right] = \omega' \sqrt{\frac{1 + V/c}{1 - V/c}} \qquad (5)$$

It is easy to see that the square root in formula (5) is larger that 1, so we see higher light frequency, the light is more "blue", that is why this frequency shift is called "blue shift". If the star moves from us, the sign of the velocity in equation (5) has to be changed to minus, and the fraction under the square root flips upside down, making the square root less than one. In this case we will see red shift. To understand that the light frequency is shifted we have to know the frequency value in the star reference frame. It turns out that all atoms emit the light at discreet, strictly defined frequencies. This applies to hydrogen and helium the most common chemical elements contained in stars. As we can see how do these frequencies shift in the light of the star, we can determine the star's velocity.

What happens if the star is moving perpendicularly to the direction from us to the star (left to right or fight to left)? In this case it can be shown that the frequency we "see" is red-shifted:

$$\omega = \omega' \sqrt{1 - V^2/c^2} \quad (6)$$

This effect is called "transverse Doppler effect". This effect does not depend on the transverse direction: both left to right motion and right to left motion lead to red shift.

Problem.

1. A spaceship moving away from the earth at a speed of 0.9c, rep[orts back from transmitting on a frequency (measured in the spaceship frame) of 100 MHz. To what frequency must Earth receivers to be tuned to receive these signals?
2. The following quote is taken from Martin Gardner's book "Relativity simply explained": "George Gamow, in one of his lectures, told a story (no doubt apocryphal) involving the Doppler effect that is much too good to be overlooked. It seems that Robert W. Wood, a famous American physicist at Johns Hopkins University, had been caught driving through a red light in Baltimore. In his appearance before the judge, Wood gave a brilliant account of the Doppler effect, explaining how his motion toward the red light had shifted the color toward the violet end of the spectrum, causing him to see it as green. The judge was set to waive the fine, but one of Wood's students (whom Wood had recently flunked) happened to be present. He pointed out the speed that would be required in order to shift the traffic light from red to green. The judge dropped the original charge, and fined Wood for speeding."

   Taking the red light wavelength as 650nm and the green light wavelength of 550nm, calculate the speed of prof. Wood's car as it was moving toward the street light.