MATH 10
ASSIGNMENT 9: MATRICES
NOV 20, 2022

Review: Vector Spaces, Basis, Dimension

Recall that a *real vector space* is a set V together with two operations: vector addition and multiplication by real numbers, satisfying natural laws. Basic example is space $\mathbb{R}^n$.

A set of vectors $\mathbf{e}_1, \dots, \mathbf{e}_n \in V$ is called a *basis* is every vector $v \in V$ can be uniquely written as linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$:

$$v = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n, \qquad x_i \in \mathbb{R}.$$

Thus, once we have chosen a basis in V, we can describe any vector by a collection of numbers $x_1, \dots, x_n$ — coordinates of this vector in our basis. For example, in $\mathbb{R}^n$ one has standard basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad , \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n$$

Important: the same vector space can have many different bases, and coordinates of a vector depend on the choice of basis — see problem 1 below. However, it is known that all bases have the same number of basis elements; this number is called *dimension* of the vector space.

Linear Maps and Matrices

Recall that an $m \times n$ matrix is just a rectangular array of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

We will denote set of all $m \times n$ matrices by $Mat_{m\times n}$.

We have used matrices before to describe a system of linear equations. Another, closely related use is for describing maps (functions) between vector spaces.

Given a matrix $A \in Mat_{m\times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, define their product $A\mathbf{x}$ by

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n \end{bmatrix}$$

Note that the result is a vector with m components; thus, an $m \times n$ matrix gives a map $\mathbb{R}^n \to \mathbb{R}^m$. It is easy to see that this map is linear:

$$A(c\mathbf{x}) = c \cdot A\mathbf{x}, \qquad c \in \mathbb{R}$$
$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$

(in fact, any linear map $\mathbb{R}^n \to \mathbb{R}^m$ is described by a matrix).

Using this notation, we can write any system of linear equations, with m equations and n variables

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1\\
&\dots\\
&a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m
\end{aligned}$$

in a compact form:

$$A\mathbf{x} = \mathbf{b}.$$

We can now restate one of the results from the last time as follows.

Theorem 1. *Let $V \subset \mathbb{R}^n$ be the set of solutions of a system of linear equations $A\mathbf{x} = 0$. Then V is a vector space, and its dimension is given by*

$$\dim V = n - r$$

where r is the number of non-zero rows in the reduced echelon form of A.

As an immediate corollary, we see that if $m < n$ (and thus $r < n$), then such a system always has non-zero solutions.

Homework

1. Show that vectors

$$\mathbf{e}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$$

form a basis in $\mathbb{R}^2$: every vector $v = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ can be uniquely written in the form

$$v = x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2$$

Can you express x_1', x_2' in terms of x_1, x_2? conversely, can you express x_1, x_2 in terms of x_1', x_2'?

2. (a) Consider the following 3 vectors in $\mathbb{R}^2$:

$$\mathbf{e}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix}3\\-4\end{bmatrix}.$$

Show that there is a linear relation between them: one can find real numebrs c_1, c_2, c_3 (not all zero) such that

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = 0.$$

(b) Deduce from the previous part that these 3 vectors do not form a basis: vector 0 can be written as combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in more than one way.

(c) Use corollary to Theorem 1 above to show that in fact, for any 3 vectors in $\mathbb{R}^2$, there must be a linear relation between them. [Hint: $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = 0$ is equivalent to a system of linear equations on c_1, c_2, c_3.]

*(d) Can you state and prove similar statement for $\mathbb{R}^n$?

3. Assume that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a basis in $\mathbb{R}^3$ (not necessarily the standard one), with the additional property that each vector has unit length and they are all orthogonal to each other:

$$|\mathbf{e}_i| = 1, \qquad \mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ for } i \neq j$$

Since it is a basis, each vector can be written in the form $v = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$.

Prove that then one can find the coordinates x_1, x_2, x_3 in this basis very easily:

$$x_i = v \cdot \mathbf{e}_i$$

4. (a) Let R be the operation of counterclockwise rotation by 90 degrees in the plane $\mathbb{R}^2$. Show that it can be described by a matrix:

$$R\begin{bmatrix}x_1\\x_2\end{bmatrix} = A\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

for some 2×2 matrix A.

(b) Can you do the same, but for the operation R_θ of rotation by angle θ? [Hint: let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis in $\mathbb{R}^2$. Can you find $R_\theta \mathbf{e}_1$, $R_\theta \mathbf{e}_2$? Once you do that, finding $R_\theta(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$ should be easy]

(c) Can you do the same for the operation of rotation by angle θ around x-axis in $\mathbb{R}^3$?

5. Solve the system of linear equations

$$\begin{aligned} y - 3z &= -1 \\ x - y + 2z &= 3 \\ 2x - y + 4z &= 8 \end{aligned}$$