

MATH 10
ASSIGNMENT 19: PERMUTATIONS
MARCH 5, 2023

A **permutation** of some set S is a function $f: S \rightarrow S$ which is a bijection (one-to-one and onto, or invertible). We will only be discussing permutations of finite sets, usually the set $S = \{1, \dots, n\}$. In this case one can also think of a permutation as a way of permuting n items placed in boxes labeled $1, \dots, n$: namely, move item from box 1 to box $f(1)$, item from box 2 to $f(2)$, etc. The set of all permutations of $\{1, \dots, n\}$ is denoted by S_n .

Permutations can be composed in the usual way: $f \circ g(x) = f(g(x))$.

Notation: the permutation f which sends 1 to a_1 , 2 to a_2 , etc, is usually written as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

A **cycle** $(a_1 a_2 \dots a_k)$ is a permutation which sends a_1 to a_2 , a_2 to a_3 , \dots , a_n to a_1 (and leaves all other elements unchanged). For example, (123) is the permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(a) = a$ for all other a .

1. How many permutations of the set $\{1, \dots, n\}$ are there?
2. Compute the following compositions (a) $(12) \circ (13)$ (b) $(12) \circ (23)$ (c) $(23) \circ (12)$
(d) $(12) \circ (13) \circ (12)$ (e) $(123) \circ (132)$ (f) $(38) \circ (123456) \circ (38)$
3. Find the inverse of permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 2 & 5 \end{pmatrix}$$

4. Fifteen students are meeting in a classroom which has 15 chairs numbered 1 through 15. The teacher requires that every minute they change seats following this rule:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	5	10	8	11	14	15	6	13	1	4	9	7	2	12

(e.g., the student who was sitting in the chair number 1 would move to chair number 3).
In how many minutes will the students return to their original seats?
5. (a) Let a permutation f be a product of non-intersecting cycles of lengths n_1, n_2, \dots, n_l (in this case, we will say that it has the **type** $\langle n_1, n_2, \dots, n_l \rangle$). What is the order of f , i.e. the smallest d such that $f^d = id$, where id is the identity permutation: $id(a) = a$?
(b) Find permutations of the set $\{1, \dots, 9\}$ which have orders 7, 10, 12, 11.
6. (a) Write the permutations in problems 3, 4 as products of non-intersecting cycles.
(b) Show that any permutation can be written as a product of non-intersecting cycles.
7. (a) Show that any permutation can be written as a product of transpositions (i.e., a permutation that interchanges two elements leaving all other unchanged — same as a cycle of length 2)
(b) Show that any permutation can be written as a product of transpositions of the form $(i \ i + 1)$
8. (a) Write the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ as a product of transpositions of the form $(i \ i + 1)$ in 2 different ways.
(b) Can you write it as a product of even number of transpositions?