MATH 6: HANDOUT 14
GEOMETRIC SEQUENCES CONTINUED

Sum of a Geometric Sequence

Let's try to sum $1 + 2 + 4 + \cdots + 64$. For purposes of working with this sum, let it be called S, i.e. $S = 1+2+4+\cdots+64$. Then I can notice that $2S = 2+4+8+\cdots+128$; subtract the original sum to get $2S-S = 128-1$ (everything else cancels out). Thus $S = 127$. What did we do here? We multiplied by 2, which lined up the terms of the sequence to the next term over. In the geometric sequence 1, 2, ..., 64, the common ratio is $q = 2$.

Let's do this in general. Let $a_1, \dots, a_n$ be a geometric sequence with common ratio q.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{a_1(1-q^n)}{1-q}$$

Proof: To prove this, we write the sum and multiply it by q:

$$S_n = a_1 + a_2 + \cdots + a_n$$
$$qS_n = qa_1 + qa_2 + \cdots + qa_n$$

Now notice that $qa_1 = a_2, \dots qa_2 = a_3, \dots, qa_n = a_{n+1}$, etc, so we have:

$$S_n = a_1 + a_2 + \cdots + a_n$$
$$qS_n = a_2 + a_3 + \cdots + a_{n+1}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$S_n - qS_n = (a_1 - a_{n+1}), \text{ or}$$
$$S_n(1-q) = (a_1 - a_1q^n)$$
$$S_n(1-q) = a_1(1-q^n)$$

from which we get the formula above.

Infinite Sum

If $0 < q < 1$, then the sum of the geometric progression is approaching some numbers, which we can call a **sum of an infinite geometric progression**, or just an **infinite sum**.

For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

The formula for the infinite sum is the following:

$$S = \frac{a_1}{1-q}$$

PROBLEMS

1. Calculate:

$$S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{10}}.$$

2. Calculate:

$$S = 1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \dots - 2^{15}$$

3. Calculate:

$$1 + x + x^2 + x^3 + x^4 + \dots + x^{100}$$

4. Calculate

$$S = 1 + 3 + 9 + 27 + 81 + 243,$$

first via the method of multiplying by the common ratio, then by plugging into the formula directly. Which method do you like better?

5. Calculate

$$S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}.$$

6. A geometric progression has 99 terms, the first term is 12 and the last term is 48. What is the 50th term?

7. If we put one grain of wheat on the first square of a chessboard, two on the second, then four, eight, $\dots$, approximately how many grains of wheat will there be? (you can use an approximation $2^{10} = 1024 \approx 10^3$).

Can you estimate the total volume of all this wheat and compare with the annual wheat harvest of the US, which is about 2 billion bushels. (A grain of wheat is about 10 mm^3; a bushel is about 35 liters, or 0.035 m^3)

***8.** Musicians use special notations for notes, i.e. sound frequencies. Namely, they go as follows:

$\dots$, A, A♯, B, C, C♯, D, D♯, E, F, F♯, G, G♯, A, A♯, $\dots$

The interval between two notes in this list is called a **halftone**; the interval between A and the next A (or B and next B, etc.) is called an **octave**. Thus, one octave is 12 halftones. (If you have never seen it, read the description of how it works in Wikipedia.)

It turns out that the frequencies of the notes above form a geometric (not an arithmetic!!) sequence: if the frequency, say, of A in one octave is 440 hz, then the frequency of A♯ is $440r$, frequency of B is $440r^2$, and so on.

(a) It is known that moving by one octave doubles the frequency: if the frequency of A in one octave is 440 hz, then the frequency of A in the next octave is $2 \times 440 = 880$ hz. Based on that, can you find the common ratio r of this geometric sequence?

(b) Using the calculator, find the ratio of frequencies of A and E (such an interval is called a **fifth**). How close is it to $3 : 2$?

Historic reference: the above convention for note frequencies is known as equal temperament and was first invented around 1585. However, it was not universally adopted until the beginning of 19th century. One of the early adopters of this tuning method was J.-S. Bach, who composed in 17221742 a collection of 48 piano pieces for so tuned instruments, called Well-Tempered Clavier. Find them and enjoy! If you want to know how musical instruments were tuned before that, do your own research.]