MATH 6: HANDOUT XXI
EUCLIDEAN GEOMETRY

Euclidean geometry tries to describe geometric properties of various figures in the plane. Figures are understood as sets of points; we will use capital letters for points and write $P \in m$ for "point P lies in figure m", or "figure m contains point P". The notion of "point" can not be defined: it is so basic that it is impossible to explain it in terms of simpler notions. In addition, there are some other basic notions (lines, distances, angles) that can not be defined. Instead, we can state some basic properties of these objects; these basic properties are usually called "postulates" or "axioms of Euclidean geometry". **All results in Euclidean geometry should be proved by deducing them from the axioms**; justifications "it is obvious", "it is well-known", or "it is clear from the figure" are not acceptable. We allow the use of all logical rules. I assume that you are familiar with some basic logical reasoning, in particular with indirect proof (also known as proof by contradiction): if assumption A leads to a contradiction, it means that A must be false. We will also use all the usual properties of numbers, equations, inequalities, etc.

For your enjoyment, take a look at the book which gave rise to Euclidean geometry and much more, Euclid's *Elements*, dated about 300 BC and used as the standard textbook for the next 2000 years. Nowadays it is available online at `http://math.clarku.edu/~djoyce/java/elements/toc.html`

1. Basic objects

These objects are the basis of all our constructions: all objects we will be discussing will be defined in terms of these objects. No definition is given for these basic objects.

- Points
- Lines
- Distances: for any two points A, B, there is a non-negative number AB, called distance between A, B.
- Angle measures: for any angle $\angle ABC$, there is a real number $m\angle ABC$, called the measure of this angle (more on this later).

We will also frequently use the words "between" when describing the relative position of points on a line (as in: A is between B and C) and "inside" (as in: point C is inside angle $\angle AOB$).

Having these basic notions, we can now define more objects. Namely, we can give definitions of

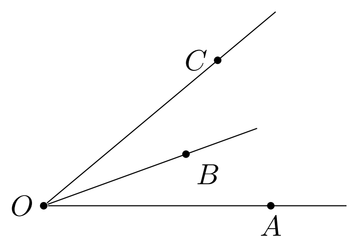
- interval, or line segment (notation: $\overline{AB}$)
- ray (notation: $\overrightarrow{AB}$)
- angle (notation: $\angle AOD$)
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points

2. First postulates

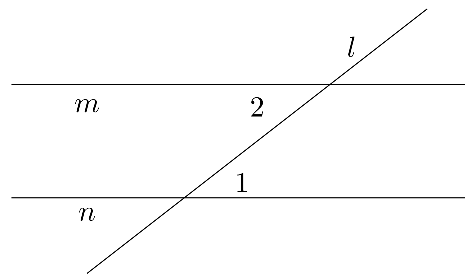
Axiom 1. For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).

Axiom 2. If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$

Axiom 3. If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.



Axiom 4. Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.



3. First theorems

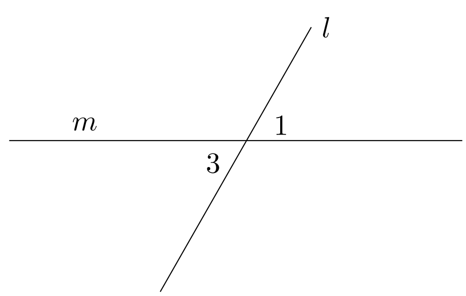
Theorem 1. If lines l, m intersect, then they intersect at exactly one point.

Proof. Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more then one point) must be false. $\square$

Theorem 2. If $l \parallel m$ and $m \parallel n$, then $l \parallel n$

Theorem 3. Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.

Proof. Let angle 2 be as shown in the figure above. Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. $\square$



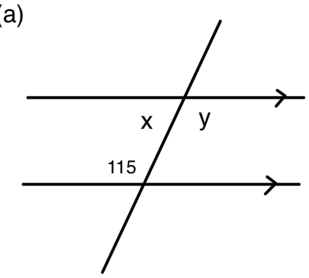
Theorem 4. Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)

Theorem 5. Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

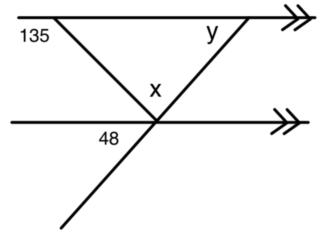
4. HOMEWORK

1. Prove Theorem 2.[Hint: draw a line that intersects the three lines and use the resulting angles to draw a conclusion.]
2. Prove Theorem 4.
3. Prove Theorem 5.
4. Find $m\angle x$ and $m\angle y$ in the following diagrams:

(a)



(b)



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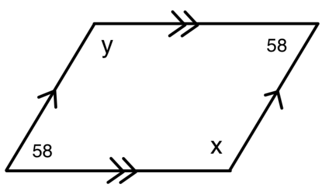

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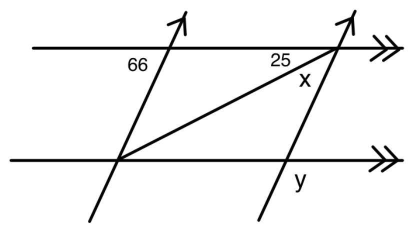
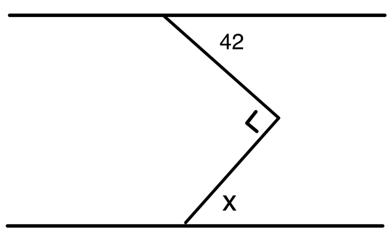


5. Find $m\angle x$ in the following diagrams:

(a)



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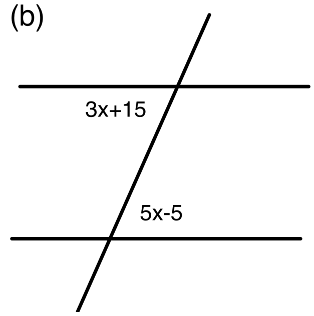
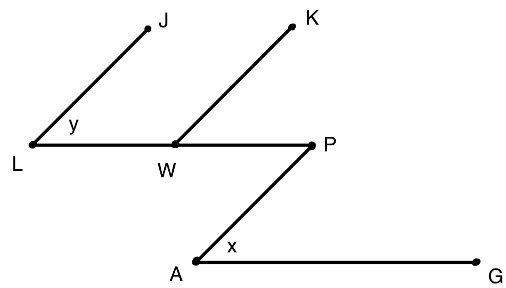
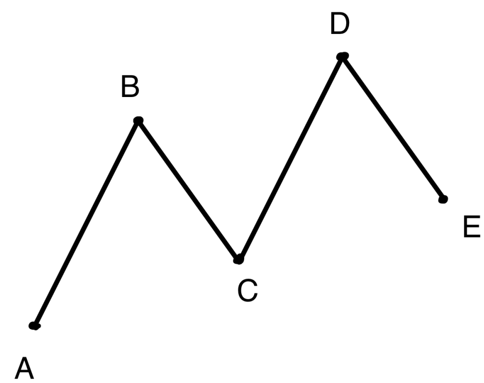


6. Assuming that $\overline{LJ} \parallel \overline{WK} \parallel \overline{AP}$ and that $\overline{PL} \parallel \overline{AG}$ in the following figure, prove that $m\angle x = m\angle y$



7. Given that $m\angle B = m\angle D$ and that $\overline{BA} \parallel \overline{DC}$, prove the $\overline{BC} \parallel \overline{DE}$.



8. Suppose we draw k lines on the plane so that each of them intersects the other, and all intersection points are distinct. Into how many pieces will they cut the plane? [Hint: how does the number of pieces change when you increase k by 1, i.e. add one more line?]