MATH 6: HANDOUT XXII
EUCLIDEAN GEOMETRY (CONTINUED)

1. Parallel and perpendicular lines

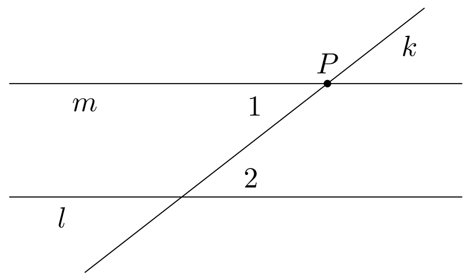
Theorem 6. Given a line l and point P not on l, there exists exactly one line m through P which is parallel to l.

Proof. **Existence.**

Let us draw a line k through P which inersects l. Now draw a line m through P such that alternate interior angles are equal: $m\angle 1 = m\angle 2$. Then, by Axiom 4 (alternate interior angles), we have $m \parallel l$.

Uniqueness.

To show that such a line is unique, let us assume that there are two different lines, m_1, m_2 through P both parallel to l. By Theorem 2, this would imply $m_1 \parallel m_2$. This gives a contradiction, because P is on both lines, but parallel lines cannot have any points in common, by definition!



□

Theorem 7. Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.

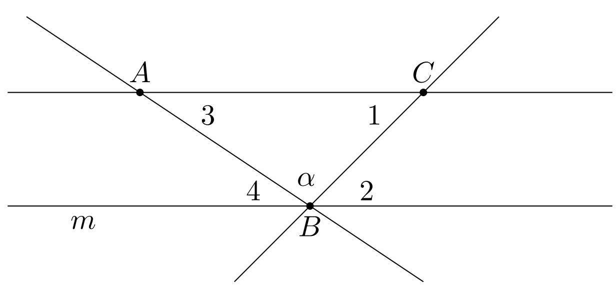
2. Sum of angles of a triangle

Definition. A triangle is a figure consisting of three distinct points A, B, C (called vertices) and line segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. We denote such a triangle by $\triangle ABC$.

Similarly, a quadrilateral is a figure consisting of 4 distinct points A, B, C, D and line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ such that these segments do not intersect except at A, B, C, D.

Theorem 8. The sum of measures of angles of a triangle is $180°$.

Proof. Draw a line m through B parallel to $\overleftrightarrow{AC}$ (possible by Theorem 6).



[By the way: α is a Greek letter, pronounced “alpha”; mathematicians commonly use Greek letters to denote angles]

Then $m\angle 2 = m\angle 1$ as alternate interior angles, and $m\angle 4 = m\angle 3$, also alternate interior angles. On the other hand, by Axiom 3 (angles add up), we have

$$m\angle 4 + m\angle\alpha + m\angle 2 = 180°$$

Thus, $m\angle 3 + m\angle\alpha + m\angle 1 = 180°$. □

Theorem 9. For a triangle $\triangle ABC$, let D be a point on continuation of side AC, so that C is between A and D. Then $m\angle BCD = m\angle A + m\angle B$. (Such an angle is called the exterior angle of triangle ABC.)

Theorem 10. Sum of angles of a quadrilateral is equal to $360°$.

3. Polygons

The total sum of the angles of a polygon is dependent on the number of sides. The sum of the exterior angles of a polygon is always 360 and does not depend on the number of sides of the polygon.

Theorem 11. The sum of the measures of the interior angles of a polygon having n sides is $180(n-2)$.

Theorem 12. The sum of the measures of the exterior angles of a polygon add up to 360.

A polygon is regular if all its sides are congruent. All its angles will also be congruent, we also call these angles equi-angular. Once you know the sum of all its angles, can you calculate the angle of a regular polygon?

4. Congruence

In general, two figures are called congruent if the have same shape and size. We use symbol $\cong$ for denoting congruent figures: to say that M_1 is congruent to M_2, we write $M_1 \cong M_2$.

Precise definition of what "same shape and size" means depends on the figure:

- For line segments, it means that they have the same length: $\overline{AB} \cong \overline{CD}$ is the same as $AB = CD$.
- For angles, it means that they have the same measure: $\angle A \cong \angle B$ is the same as $m\angle A = m\angle B$.
- For triangles, it means that the corresponding sides are equal and corresponding angles are equal: $\triangle ABC \cong \triangle A'B'C'$ is the same as
  $AB = A'B'$, $BC = B'C'$, $AC = A'C'$, $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $m\angle C = m\angle C'$.

Note that for triangles, the notation $\triangle ABC \cong \triangle A'B'C'$ not only tells that these two triangles are congruent, but also shows which vertex of the first triangle corresponds to which vertex of the second one. For example, $\triangle ABC \cong \triangle PQR$ is not the same as $\triangle ABC \cong \triangle QPR$.

5. Congruence tests for triangles

By definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalitites. However, it turns out that in fact, we can do with fewer checks.

Axiom 5 (Angle-Side-Angle Congruence Axiom)**.** If $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ and $AB = A'B'$, then $\triangle ABC \cong \triangle A'B'C'$.

This axiom is commonly referred to as ASA axiom.

One can also try other ways to define a triangle by three pieces of information, such as three sides (SSS), three angles (AAA), or two sides and an angle. For the two sides and an angle, there are two versions, one in which the two sides are adjacent to the given angle (SAS) and the other in which one of the given sides is opposite to the given angle (SSA). It turns out that SSS and SAS do indeed define a triangle:

Axiom 6 (SSS Congruence Axiom)**.** If $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.

Axiom 7 (SAS Congruence Axiom)**.** If $AB = A'B'$, $AC = A'C'$ and $m\angle A = m\angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.

These congruence rules can actually be derived by introducing additional concepts, so we do not really need to take them as axioms. However, I would rather not spend time on this (proof of SSS takes some time), so I decided to take them as axioms.

6. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem 13 (Base angles equal)**.** If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.

Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

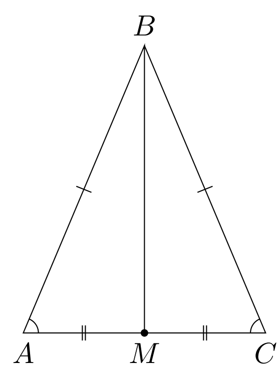
Proof. Assume that $\triangle ABC$ is isoceles, with apex B. Then by SAS, we have $\triangle ABC \cong \triangle CBA$. Therefore, $m\angle A = m\angle C$.

The proof of the converse statement is left to you as a homework exercise. □

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC (it exists and is unique by Theorem 7); the median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

Theorem 14. If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.



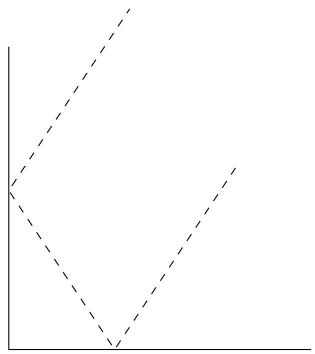
Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and $m\angle MAB = m\angle MCB$ (by Theorem 13). Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.

Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. □

1. Show that if, in a quadrilateral $ABCD$, diagonally opposite angles are equal ($m\angle A = m\angle C$, $m\angle B = m\angle D$), then opposite sides are parallel. [Hint: show first that $m\angle A + m\angle B = 180°$.]

2. The reflection law states that the angles formed by the incoming light ray and the reflected one with the surface of the mirror are equal: $m\angle 1 = m\angle 2$

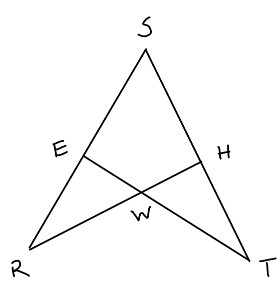
Using this law, show that a corner made of two perpendicular mirrors will reflect any light ray exactly back: the reflected ray is parallel to the incoming one:



This property – or rather, similar property of corners in 3-D – is widely used: reflecting road signs (including stop signs), tail lights of a car, and reflecting strips on clothing are all constructed out of many small reflecting corners so that they reflect the light of a car headlamp exactly back to the car.

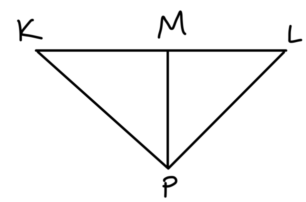
3. Given that $\angle R \cong \angle T$ and $\overline{SR} \cong \overline{ST}$, prove that $\triangle SRH \cong \triangle STE$.

4. Given that $\overline{TE} \perp \overline{RS}$, that $\overline{RH} \perp \overline{ST}$, and $\overline{EW} \cong \overline{HW}$, prove that $\triangle EWR \cong \triangle HWT$



5. Given that $\overline{PM}$ is the altitude to $\overline{KL}$ and $\overline{PK} \cong \overline{PK}$, prove that M is the midpoint of $\overline{KL}$.

6. Given that $\overline{PM}$ is the median to $\overline{KL}$ and $\overline{KP} \cong \overline{LP}$, prove that $\overline{PM} \perp \overline{KL}$.



7. Let $\triangle ABC$ be such that all sides have equal length. Prove that then $m\angle A = m\angle B = m\angle C = 60^{\circ}$. [Such a triangle is called equilateral.]

8. Prove the second statement of $Theorem\ 13$: if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

9. Let $ABCD$ be a quadrilateral such that $AB = BC = CD = AD$ (such a quadilateral is called rhombus). Let M be the intersection point of AC and BD.

(a) Show that $\triangle ABC \cong \triangle ADC$

(b) Show that $\triangle AMB \cong \triangle AMD$

(c) Show that the diagonals are perpendicular and that the point M is the midpoint of each of the diagonals.