MATH 7: CLASSWORK 6
INTRODUCTIONS TO COMBINATORICS. PASCAL TRIANGLE.
November 6, 2022

COUNTING

Fundamental Principle of Counting (Multiplication rule). If the first task can be performed in m ways, and for each of these a second task can be performed in n ways, and for each combination a third task can be performed in k ways, etc. then this entire sequence of tasks can be performed in
$m \cdot n \cdot k$ … ways.

Permutations: the choice of k things from a set of n things without repetition(“replacement”) and where the **order matters**.

- Picking first, second, and third place winners from a group. If a group has n members, then this can be done in $n(n-1)(n-2)$ ways.

Permutations of n things: The number of permutations of all n different things: **n!**

- Arranging/ordering all n members of a group can be done in n! ways.
- Listing the favorite deserts in the order of choices: if there are n desserts in total, there are **n!** ways to arrange them in the order of preference.

Combinations: the choice of k things from a set of n things without repetition (“replacement”), where we have identical items, and where **order does not matter**. Combinations are harder to count: we will talk about it later!

- Picking three team members from a group (it doesn’t matter who is chosen first, or second, or third).
- Picking two deserts from a tray (the order in which you eat them doesn’t matter!).

PASCAL TRIANGLE

How many ways are there to go from the bottom left corner of the chessboard to the upper right, moving always only to the right and up?

To make out life easier, we will refer to cells by two numbers (m, n): m is the number of the column (counting from left), and n is the number of the row (counting from the bottom).

We can solve this problem iteratively. There is only one path to any of the cells in the lowest row or the left column. Let’s put ones there. Now, let’s think about other cells.

To get to cell $(2, 2)$, we can first get to cell $(2, 1)$ (there’s only 1 way to get there), and then do a step up; or first get to cell $(1, 2)$ (there’s again only 1 way to get there), and then do a step right. That means, there are 1 + 1 = 2 ways to get to cell $(2, 2)$.

Now, let’s think about cell $(3, 2)$. Again, we have two choices: first, we can get there from cell $(2, 2)$ by doing a step right, or second, we can get there from cell (3, 1) by doing a step up. It means the total number of paths to get to cell (3, 2) will be equal to the total number of ways to get to (2, 2) plus the total number of ways to get to (3, 1): 2 + 1 = 3.

Keeping this process going, we can notice that the number of paths to cell (m, n) is equal to the number of paths to cell $(m - 1, n)$ plus the number of paths to cell $(m, n - 1)$. This way we can fill out the entire table:

1	6	21	56	126	252
1	5	15	35	70	126
1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

These numbers are called the *binomial coefficients*. They are usually written in a slightly different way:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
. . .

This triangle is called **Pascal triangle**. Every entry in it is obtained as the sum of two entries above it. The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by $_nC_k$. Note that **both n and k are counted from 0**, not from 1:
for example, $\binom{2}{1} = 2, \binom{3}{0} = 1$.