

MATH 7: HOMEWORK 12

General quadratic formula

January 15, 2023

1. Quadratic equation in a standard form.

Standard form: $ax^2 + bx + c = 0$

A quadratic equation could have

- no solution,
- one solution,
- two solutions depending on the coefficients a, b, and c.

Factored form: $(x - x_1)(x - x_2) = 0$, where x_1 and x_2 are the solutions of the equation, also known as *roots*.

2. Solving the incomplete quadratic equation by factorizing.

➤ When $c = 0$, $ax^2 + bx = 0$

$x(ax + b) = 0$ The two roots are $x_1 = 0$ and $x_2 = -b/a$

➤ When $b = 0$, $ax^2 + c = 0$

If $c < 0$, factorize the equation using the formula for fast multiplication $a^2 - b^2 = (a - b)(a + b)$. (*)

For example, $x^2 - 25 = 0 \Rightarrow x^2 - 5^2 = 0 \Rightarrow (x - 5)(x + 5) = 0$. $x = \pm 5$

If $c > 0$, there are no real solutions. An easy way to see this is to solve directly for x : $x^2 + 25 = 0 \Rightarrow x^2 = -25$; No number squared is equal to a negative number!

2. Solving the complete quadratic equation

➤ By completing the square

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x + 3)^2 - 7 = (x + 3)^2 - (\sqrt{7})^2 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

Thus, $x^2 + 6x + 2 = 0$ if and only if $(x + 3 + \sqrt{7}) = 0$, which gives $x = -3 - \sqrt{7}$, or $(x + 3 - \sqrt{7}) = 0$, which gives $x = -3 + \sqrt{7}$.

➤ By using the quadratic formula

Completing the square works in general for any quadratic equation in a standard form

If $a = 1$, then:

$$x^2 + bx + c = x^2 + 2 \frac{b}{2}x + c = \left(x^2 + 2 \frac{b}{2}x + \frac{b^2}{2^2}\right) - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{2^2} = \left(x + \frac{b}{2}\right)^2 - \frac{D}{2^2} \quad \text{eq (1)}$$

$$\text{Thus } x^2 + bx + c = 0 \text{ is equivalent to: } \left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$$

If $a \neq 1$, then: $ax^2 + bx + c = 0$ divide by $a \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x^2 + 2 \frac{b}{2a}x + \frac{b^2}{2^2 a^2}\right) - \frac{b^2}{2^2 a^2} + c = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{2^2 a^2}$$

is equivalent to: $\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$, where $D = b^2 - 4ac$

The determinant D determines the number of solutions. $D < 0$, there are no real solutions; if $D = 0$, there is one solution,

if $D > 0$, the solutions are:

$$x + \frac{b}{2a} = \pm \sqrt{\frac{D}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

$$D = b^2 - 4ac$$

eq (2)

Homework problems

Instructions: Please always write solutions on a **separate sheet of paper**. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

Note: Use the formulas for fast multiplication $a^2 - b^2 = (a - b)(a + b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

- Complete the square and solve the quadratic equations: using $(a \pm b)^2 = a^2 \pm 2ab + b^2$
and then $a^2 - b^2 = (a - b)(a + b)$
 - $x^2 - 2x - 3 = 0$
 - $x^2 + 8x - 9 = 0$
- Solve the following equations. Carefully think what method you will use and write all steps in your solution. The following questions may help you: is the equation in a standard or in a factored form?; what are the coefficients a, b, c? Are some of these coefficients zero? Shall I factorize or use the quadratic formula from eq (2)?
 - $x^2 - 5x + 5 = 0$
 - $x^2 = 1 + x$
 - $-4x^2 + 8x + 21 = 0$
 - $2x(3 - x) = 1$
 - $x^3 + 4x^2 - 45x = 0$
 - $\frac{x}{x-2} = x - 1$
- Indian mathematicians were aware of the quadratic formula for solving quadratic equations. Can you solve the following problem by the 9th century mathematician Mahavira? (translated from original text)

One-third of a herd of elephants and three times the square root of the remaining part of the herd were seen on a mountain slope; and in a lake was seen a male elephant along with three female elephants constituting the ultimate remainder. How many were the elephants here?
- In the 12th century, Indian mathematician Bhaskara formulated the following problem. Solve it! (translated from original text)

Out of a party of monkeys, the square of one fifth of their number diminished by three went into a cave. The one remaining monkey was climbing up a tree. What is the total number of monkeys?
- Use eq (2) to solve these equations:
 - $4x^2 - 58 + 5 = 0$
 - $2x^2 + 5x + 3 = 0$
- Determine the number of solutions of the following equations. You do not need to solve them!
 - $2x^2 + 5x - 1 = 0$
 - $3x^2 - 4x + 10 = 0$
 - $3x^2 - 24x + 48 = 0$
 - $5x^2 + 7x + 6 = 0$
- If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2}$ and $x^3 + \frac{1}{x^3}$ [Hint: try completing the square, completing the cube ...]
- Solve equation $x = \frac{x}{2} + \frac{1}{x}$; [Hint: multiply by $2x$]