MATH 7: HOMEWORK 13
Vieta's formulas
January 22, 2023

1. Solving the complete quadratic equation by smart guessing

If x_1 and x_2 are the roots, then the quadratic equation can be written from standard into factored form:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

For the special case when a = 1, we set a = 1 and then multiplying the expressions in the right hand side, we get: $x^2 + bx + c = (x - x_1)(x - x_2) = x^2 - x_1 - x_2x + x_1x_2 = x^2 - (x_1 + x_2)x + \boldsymbol{x_1x_2}$

From comparing the numbers in front of the same powers of x the coefficients we can get the following:

$$x_1 + x_2 = -b$$
$$x_1x_2 = c$$

- **for equation in standard form:** $ax^2 + bx + c = 0$

The roots of the quadratic equation are related to the coefficients: $\boldsymbol{x_1x_2 = \frac{c}{a}}$ and $\boldsymbol{x_1 + x_2 = -\frac{b}{a}}$

In the special case when $\boldsymbol{a = 1}$, $\boldsymbol{x_1x_2 = \mathrm{c}}$ and $\boldsymbol{x_1 + x_2 = -b}$

$$\boxed{\begin{array}{c} \boldsymbol{x^2 + bx + c = 0} \\ \boldsymbol{x_1x_2 = \mathrm{c}, \quad x_1 + x_2 = -b} \end{array}}$$

Homework problems

Instructions: Please always write solutions on a ***quadrille sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Find the roots of the equations x_1 and x_2 using Vieta's formulas. After that, write the equation in a factorized form as $a(x - x_1)(x - x_2)$.
   a. $x^2 - 5x + 6 = 0$
   b. $x^2 + 8x - 9 = 0$
   c. $2x^2 + 4x - 6 = 0$

2. Let x and y be some numbers. Use the formulas for fast multiplication to rewrite the following expressions using only $(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{B}$ and $\boldsymbol{xy} = \boldsymbol{C}$, where B and C are just number. To do that, present the expressions as sums and/or products of x and y, then substitute the sums/products with B and C. No x and y are allowed in the answers!

Example: $x^2 + y^2 = x^2 + y^2 + 2xy - 2xy = (x+y)^2 - 2xy = B^2 - 2C$

We completed the square using the formula: $a^2 \pm 2ab + b^2 = (a \pm b)^2$

   a. $(x - y)^2 =$
   b. $\frac{1}{x} + \frac{1}{y} =$
   c. $\frac{1}{x-1} + \frac{1}{y-1} =$
   d. (*) $x^3 + y^3$

   Hint: first compute $(x + y)(x^2 + y^2)$

3. Let x_1 , x_2 be the roots of the equation $x^2 + 5x - 7 = 0$. Using the Vieta's formulas, find the values of the expressions *without* explicitly calculating x_1 and x_2
   a. ${x_1}^2 + {x_2}^2 =$
   b. $(x_1 - x_2)^2 =$
   c. $\frac{1}{x_1} + \frac{1}{x_2} =$
   d. (*) ${x_1}^3 + {x_2}^3 =$

4. Solve the following equations.
   a. $x^2 - 5x + 6 = 0$
   b. $5x^2 = 8x - 3$
   c. $\sqrt{2x + 1} = x$
   d. $x + \frac{2}{x} = 3$

5. Solve the biquadratic equation: $x^4 - 3x^2 + 2 = 0$

6. Prove the following inequalities:
   a. for any $a > 0$, prove that $a + \frac{1}{a} \geq 2$, with equality only when a = 1.
   b. Show that for any $a, b \geq 0$, one has

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

   The left hand side is usually called the arithmetic mean of a, b; the right hand side is called the geometric mean of a, b .

Hint: I suggest you "remove" the denominator or the radicals (square roots), write the quadratic equations into factorized form, with zero on one side of the new inequality. Try your best, we will be solving inequalities soon.