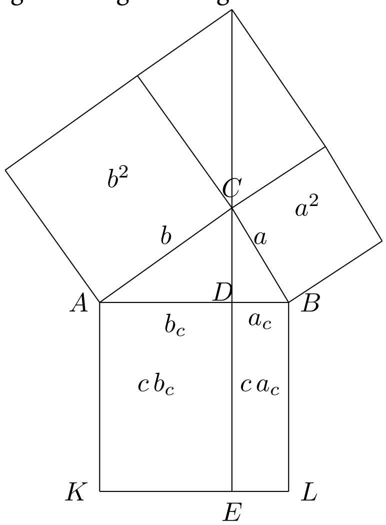
MATH 7: HANDOUT 5
SQUARE TRIANGLES, PYTHAGORAS THEOREM, INCOMMENSURATE LENGTHS

Pythagoras' Theorem

In a right triangle with legs a and b, and hypotenuse c, the square of the hypotenuse is the sum of squares of each leg. $c^2 = a^2 + b^2$. The converse is also true, if the three sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.



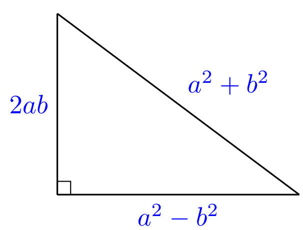
In the picture, areas of rectangles $AKED$ and $BLED$ can be shown

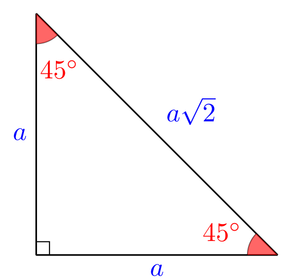
$$S_{AKED} = c \cdot b_c = b^2\,,$$
$$S_{BLED} = c \cdot a_c = a^2\,,$$

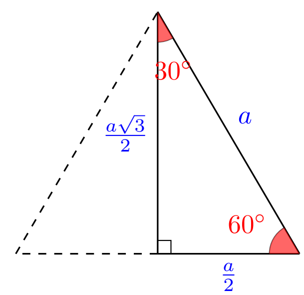
thus $a^2 + b^2 = c^2$
Also, equalities of the areas give important formulas

$$|AB| \cdot |AD| = |AC|^2\,,$$
$$|AB| \cdot |BD| = |BC|^2\,,$$

Some Pythagorean triples (a, b, c) are
(3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (9, 40, 41), (11, 60, 61), (20, 21, 29).
To generate such Pythagorean triples, choose two positive integers a and b and plug the values into the sides as shown on the first picture (try figure out why the sides of this triangle satisfy the Pythagoras' Theorem!)



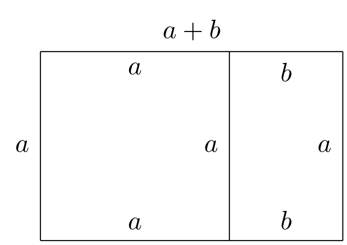




Other notable triangles are
45-45-90 Triangle: If one of the anglesin a right triangle is $45°$, the other angle is also $45°$, and two of its legs are equal. If the length of a leg is a, the hypothenuse is $a\sqrt{2}$.
30-60-90 Triangle: If one of the angles in a right triangle is $30°$, the other angle is $60°$. Such triangle is a half of the equilateral triangle. That means that if the hypothenuse is equal to a, its smaller leg is equal to the half of the hypothenuse, i.e. $\frac{a}{2}$. From the Pythagoras' Theorem, the other leg is $\frac{a\sqrt{3}}{2}$.

Thanks to Pythagoras' theorem, one can construct "incommensurate" segments like $\sqrt{2}a$, $\sqrt{3}a$, etc. For example, the golden ratio ("Golden section") is widespread in the art and architecture,

$$\frac{a+b}{a} = \frac{a}{b} \quad \Leftrightarrow \quad \frac{a}{b} = \frac{\sqrt{5}+1}{2} = \frac{2}{\sqrt{5}-1}$$



Homework

1. Using algebraic identities calculate

(a) $299^2 + 598 + 1 =$

(b) $199^2 =$

(c) $51^2 - 102 + 1 =$

2. Write each of the following expressions in the form $a + b\sqrt{3}$, with rational a, b:

(a) $(1+\sqrt{3})^2$

(b) $(1+\sqrt{3})^3$

(c) $\dfrac{1}{1-2\sqrt{3}}$

(d) $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

(e) $\dfrac{1+2\sqrt{3}}{\sqrt{3}}$

3. In a trapezoid ABCD with bases AD and BC, $\angle A = 90°$, and $\angle D = 45°$. It is also known that $AB = 10$ cm, and $AD = 3BC$. Find the area of the trapezoid.

4. In a right triangle ABC, BC is the hypotenuse. Draw AD perpendicular to BC, where D is on BC. The length of BC=13, and AB=5. What is the length of AD?

5. What is the area of a regular hexagon whose side is 5cm?

6. Draw a segment of some length a (say a=2cm). Use a straightedge and a compass to construct segments

(a) $\sqrt{7}a$ (b) $\sqrt{11}a$ (c) $\sqrt{13}a$ (d) $\sqrt{17}a$

Mark all intermediate steps and segments!